



Simple Proof for the Riemann Hypothesis

Frank Vega

EasyChair preprints are intended for rapid dissemination of research results and are integrated with the rest of EasyChair.

November 22, 2023

Simple Proof for the Riemann Hypothesis

Frank Vega

To my mother

Abstract. Let $\Psi(n) = n \cdot \prod_{q|n} \left(1 + \frac{1}{q}\right)$ denote the Dedekind Ψ function where $q | n$ means the prime q divides n . Define, for $n \geq 3$; the ratio $R(n) = \frac{\Psi(n)}{n \cdot \log \log n}$ where \log is the natural logarithm. Let $M_x = \prod_{q \leq x} q$ be the product extending over all prime numbers q that are less than or equal to a natural number $x > 1$. The Riemann hypothesis is the assertion that all non-trivial zeros are complex numbers with real part $\frac{1}{2}$. It is considered by many to be the most important unsolved problem in pure mathematics. There are several statements equivalent to the Riemann hypothesis. In 2011, Solé and Planat stated that the Riemann hypothesis is true if and only if the inequality $R(M_x) > \frac{e^\gamma}{\zeta(2)}$ holds for all $x \geq 5$, where $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and $\zeta(x)$ is the Riemann zeta function. In this note, using Solé and Planat criterion, we prove that the Riemann hypothesis is true.

Mathematics Subject Classification (2010). Primary 11M26; Secondary 11A41.

Keywords. Riemann hypothesis, prime numbers, Riemann zeta function, Chebyshev function.

1. Introduction

The Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. It was proposed by Bernhard Riemann (1859). The Riemann hypothesis belongs to the Hilbert's eighth problem on David Hilbert's list of twenty-three unsolved problems. This is one of the Clay Mathematics Institute's Millennium Prize Problems. In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{q \leq x} \log q$$

with the sum extending over all prime numbers q that are less than or equal to x , where \log is the natural logarithm.

Proposition 1.1. *For every $x > 1$ [6, Theorem 4 (3.15) pp. 70]:*

$$\theta(x) < \left(1 + \frac{1}{2 \cdot \log x}\right) \cdot x.$$

The following property is based on natural logarithms:

Proposition 1.2. *For $x > -1$ [3, pp. 1]:*

$$\log(1 + x) \leq x.$$

Leonhard Euler studied the following value of the Riemann zeta function (1734) [1].

Proposition 1.3. *We define [1, (1) pp. 1070]:*

$$\zeta(2) = \prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1} = \frac{\pi^2}{6},$$

where q_k is the k th prime number. By definition, we have

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

where n denotes a natural number. Leonhard Euler proved in his solution to the Basel problem that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1} = \frac{\pi^2}{6},$$

where $\pi \approx 3.14159$ is a well-known constant linked to several areas in mathematics such as number theory, geometry, etc.

The number $\gamma \approx 0.57721$ is the Euler-Mascheroni constant which is defined as

$$\begin{aligned} \gamma &= \lim_{n \rightarrow \infty} \left(-\log n + \sum_{k=1}^n \frac{1}{k} \right) \\ &= \int_1^{\infty} \left(-\frac{1}{x} + \frac{1}{[x]} \right) dx. \end{aligned}$$

Here, $[\dots]$ represents the floor function. Franz Mertens discovered some important results about the constants B and H (1874) [4]. The number $B \approx 0.26149$ is the Meissel-Mertens constant where $\gamma = B + H$ [4].

Proposition 1.4. *We have [2, Lemma 2.1 (1) pp. 359]:*

$$\sum_{k=1}^{\infty} \left(\log \left(\frac{q_k}{q_k - 1} \right) - \frac{1}{q_k} \right) = \gamma - B = H.$$

For $x \geq 2$, the function $u(x)$ is defined as follows [5, pp. 379]:

$$u(x) = \sum_{q > x} \left(\log \left(\frac{q}{q-1} \right) - \frac{1}{q} \right).$$

On the sum of the reciprocals of all prime numbers not exceeding x , we have:

Proposition 1.5. For $x > 1$ [6, Theorem 5 (3.17) pp. 70]:

$$-\frac{1}{2 \cdot \log^2 x} < \sum_{q \leq x} \frac{1}{q} - B - \log \log x.$$

In number theory, $\Psi(n) = n \cdot \prod_{q|n} \left(1 + \frac{1}{q}\right)$ is called the Dedekind Ψ function where $q | n$ means the prime q divides n . For $x \geq 2$, a natural number M_x is defined as

$$M_x = \prod_{q \leq x} q.$$

We define $R(n) = \frac{\Psi(n)}{n \cdot \log \log n}$ for $n \geq 3$. We say that Dedekind(x) holds provided that

$$R(M_x) > \frac{e^\gamma}{\zeta(2)}.$$

Proposition 1.6. Dedekind(x) holds for all $x \geq 5$ if and only if the Riemann hypothesis is true [7, Theorem 4.2 pp. 5].

Putting all together yields a proof for the Riemann hypothesis.

2. Central Lemma

The following is a key Lemma.

Lemma 2.1. The inequality $\frac{\prod_{q \leq x} \left(\frac{q}{q-1}\right)}{\log \theta(x)} \geq \left(\frac{e^\gamma}{\zeta(2)}\right)^7$ holds for all $x \geq 10^8$.

Proof. By Proposition 1.4, the inequality

$$\frac{\prod_{q \leq x} \left(\frac{q}{q-1}\right)}{\log \theta(x)} \geq \left(\frac{e^\gamma}{\zeta(2)}\right)^7$$

is the same as

$$\sum_{q \leq x} \log \left(\frac{q}{q-1} \right) - B - \log \log \theta(x) \geq H + 6 \cdot \gamma - 7 \cdot \log(\zeta(2))$$

after of applying the logarithm to the both sides and distributing the terms. In addition,

$$\begin{aligned}
\log \log \theta(x) &< \log \log \left(\left(1 + \frac{1}{2 \cdot \log x} \right) \cdot x \right) \\
&= \log \left(\log \left(1 + \frac{1}{2 \cdot \log x} \right) + \log x \right) \\
&= \log \left((\log x) \cdot \left(1 + \frac{\log \left(1 + \frac{1}{2 \cdot \log x} \right)}{\log x} \right) \right) \\
&= \log \log x + \log \left(1 + \frac{\log \left(1 + \frac{1}{2 \cdot \log x} \right)}{\log x} \right) \\
&\leq \log \log x + \frac{\log \left(1 + \frac{1}{2 \cdot \log x} \right)}{\log x} \\
&\leq \log \log x + \frac{1}{2 \cdot \log^2 x}
\end{aligned}$$

by Propositions 1.1 and 1.2. So,

$$\sum_{q \leq x} \log \left(\frac{q}{q-1} \right) - B - \log \log x - \frac{1}{2 \cdot \log^2 x} \geq H + 6 \cdot \gamma - 7 \cdot \log(\zeta(2)).$$

That is,

$$\sum_{q \leq x} \log \left(\frac{q}{q-1} \right) - B - \log \log x - \frac{1}{2 \cdot \log^2 x} - u(x) \geq H - u(x) + 6 \cdot \gamma - 7 \cdot \log(\zeta(2)).$$

after subtracting $u(x)$ to the both sides of the inequality. By Proposition 1.4, we can see that

$$\sum_{q \leq x} \left(\frac{1}{q} \right) - B - \log \log x - \frac{1}{2 \cdot \log^2 x} - u(x) \geq 6 \cdot \gamma - 7 \cdot \log(\zeta(2)).$$

By Proposition 1.5, we deduce that

$$-\frac{1}{2 \cdot \log^2 x} - \frac{1}{2 \cdot \log^2 x} - u(x) \geq 6 \cdot \gamma - 7 \cdot \log(\zeta(2)).$$

It is a fact that the inequality

$$-\frac{1}{\log^2 x} - u(x) \geq 6 \cdot \gamma - 7 \cdot \log(\zeta(2))$$

holds for all $x \geq 10^8$ due to

$$-\frac{1}{\log^2 x} - H \geq 6 \cdot \gamma - 7 \cdot \log(\zeta(2)) - \sum_{q \leq 10^8} \left(\log \left(\frac{q}{q-1} \right) - \frac{1}{q} \right). \quad \square$$

3. Main Insight

This is the main insight.

Lemma 3.1. *Dedekind(x) always holds for all $x \geq 10^8$.*

Proof. By Lemma 2.1, the inequality

$$\frac{\prod_{q \leq x} \left(\frac{q}{q-1} \right)}{\log \theta(x)} \geq \left(\frac{e^\gamma}{\zeta(2)} \right)^7$$

holds for all $x \geq 10^8$. By Propositions 1.2 and 1.4, the inequality

$$\frac{\prod_{q \leq x} \left(\frac{q}{q-1} \right)}{\log \theta(x)} \geq \left(\frac{e^\gamma}{\zeta(2)} \right)^7$$

is equivalent to

$$e^{H-u(x)} \cdot R(M_x) \geq \left(\frac{e^\gamma}{\zeta(2)} \right)^7.$$

Certainly, we have

$$\begin{aligned} \frac{\prod_{q \leq x} \left(\frac{q}{q-1} \right)}{\log \theta(x)} &\geq \left(\prod_{q \leq x} \frac{\left(\frac{q}{q-1} \right)}{e^{\frac{1}{q}}} \right) \cdot \frac{\prod_{q \leq x} \left(1 + \frac{1}{q} \right)}{\log \theta(x)} \\ &= e^{H-u(x)} \cdot \frac{\prod_{q \leq x} \left(1 + \frac{1}{q} \right)}{\log \theta(x)} \\ &= e^{H-u(x)} \cdot \frac{M_x \cdot \prod_{q|M_x} \left(1 + \frac{1}{q} \right)}{M_x \cdot \log \log M_x} \\ &= e^{H-u(x)} \cdot \frac{\Psi(M_x)}{M_x \cdot \log \log M_x} \\ &= e^{H-u(x)} \cdot R(M_x) \end{aligned}$$

using the Propositions 1.2 and 1.4 such that $e^{\frac{1}{q}} \geq \left(1 + \frac{1}{q} \right)$ for every prime q . Consequently, we would have

$$\frac{e^{H-u(x)}}{\left(\frac{e^\gamma}{\zeta(2)} \right)^6} \cdot R(M_x) \geq \frac{e^\gamma}{\zeta(2)}.$$

We only need to prove that

$$\frac{e^{H-u(x)}}{\left(\frac{e^\gamma}{\zeta(2)} \right)^6} < 1$$

holds for all $x \geq 10^8$ to confirm that *Dedekind(x)* also holds. Hence, it is enough to show that

$$\frac{e^H}{\left(\frac{e^\gamma}{\zeta(2)} \right)^6} < 1$$

because of

$$e^{u(x)} > 1$$

for $x \geq 2$ [5, (11) pp. 379]. Next,

$$-B + 6 \cdot \log(\zeta(2)) - 5 \cdot \gamma < 0$$

after applying the logarithm to the both sides of the inequality. Finally, we obtain that

$$\frac{6}{5} < \frac{\gamma + \frac{B}{5}}{\log(\zeta(2))}.$$

Using a simple numerical calculation, we can check that

$$\frac{\gamma + \frac{B}{5}}{\log(\zeta(2))} > 1.26483 > 1.2 = \frac{6}{5}$$

and therefore, the proof is done. \square

4. Main Theorem

This is the main theorem.

Theorem 4.1. *The Riemann hypothesis is true.*

Proof. We already know that $\text{Dedekind}(x)$ holds for all $5 \leq x \leq 10^8$ [5, Theorem 3 (a) pp. 376]. In this way, the Riemann hypothesis must be true as a direct consequence of the Proposition 1.6 and Lemma 3.1. \square

References

- [1] Ayoub, R.: Euler and the Zeta Function. *The American Mathematical Monthly* **81**(10), 1067–1086 (1974). <https://doi.org/10.2307/2319041>
- [2] Choie, Y., Lichiardopol, N., Moree, P., Solé, P.: On Robin’s criterion for the Riemann hypothesis. *Journal de Théorie des Nombres de Bordeaux* **19**(2), 357–372 (2007). <https://doi.org/10.5802/jtnb.591>
- [3] Kozma, L.: Useful Inequalities. Kozma’s Homepage, Useful inequalities cheat sheet. http://www.lkozma.net/inequalities_cheat_sheet/ineq.pdf (2011), Accessed 17 November 2023
- [4] Mertens, F.: Ein Beitrag zur analytischen Zahlentheorie. *J. reine angew. Math.* **1874**(78), 46–62 (1874). <https://doi.org/10.1515/crll.1874.78.46>
- [5] Nicolas, J.L.: Petites valeurs de la fonction d’Euler. *Journal of Number Theory* **17**(3), 375–388 (1983). [https://doi.org/10.1016/0022-314X\(83\)90055-0](https://doi.org/10.1016/0022-314X(83)90055-0)
- [6] Rosser, J.B., Schoenfeld, L.: Approximate Formulas for Some Functions of Prime Numbers. *Illinois Journal of Mathematics* **6**(1), 64–94 (1962). <https://doi.org/10.1215/ijm/1255631807>
- [7] Solé, P., Planat, M.: Extreme values of the Dedekind ψ function. *Journal of Combinatorics and Number Theory* **3**(1), 33–38 (2011)

Frank Vega
NataSquad
10 rue de la Paix
FR 75002 Paris
France
e-mail: vega.frank@gmail.com