2 x 2 Integer Matrices: Composition of Binary Quadratic Forms

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ABSTRACT

In this research paper, we consider 2 x 2 integer matrices and identify interesting binary quadratic forms which naturally arise. Specifically, we consider such symmetric integer matrices and derive compositions of pure binary quadratic forms naturally arising in association with determinant of such matrices. We also, discover number-theoretic results associated with trinary quadratic forms naturally arising in connection with 2 x 2 symmetric integer matrices. We formulate a "generalized Waring problem" using real quadratic algebraic numbers. We also discuss composition of binary quadratic forms naturally arising in other interesting structured 2 x 2 integer matrices. We explore representation of integers using trinary as well as binary quadratic forms.

1. INTRODUCTION:

Ever since the dawn of civilization, integers stimulated the curiosity of several mathematicians. Algebraic symbolism helped defining negative numbers based on the concept of "ZERO". Also, linear algebraic equation in one variable with integer coefficients enabled the introduction of rational numbers. Similarly, quadratic equations naturally led to the proposal of novel class of numbers, called "complex numbers".

Diophantus considered linear algebraic equations in two variables that are constrained to be integers. The so called simplest linear Diophantine equation is of the form

$$a x + b y = c$$

was solved using the Greatest Common Divisor (GCD of the integers { a, b }) algorithm. The book "Arithmetica" by Diophantus and some volumes of Euclid's elements contained interesting results related to integers and specifically prime numbers, perfect numbers (among other number-theoretic results). Fermat acquired a copy of Arithmetica and contributed several interesting number-theoretic theorems that survived passage of time. For instance, Fermat proved that a prime number of the form { 4 l + 1, l=1,2,… } (i.e. $p \equiv 1\ (mod)\ 4$) can be expressed uniquely as the sum of squares of two integers. Further, a prime of the form { 4 l + 3, l=1,2,…….. } (i.e. $p \equiv 1\ (mod)\ 4$) can never be expressed as the sum of squares of two integers. This result was combined with the following algebraic identity

$$\begin{aligned}(x_1^2 + x_2^2)(y_1^2 + y_2^2) &= (x_1 y_1 - x_2 y_2)^2 + (x_1 y_2 + x_2 y_1)^2 \\ &= (x_1 y_1 + x_2 y_2)^2 + (x_1 y_2 - x_2 y_1)^2\end{aligned}$$

resulting in the so called GENUS theorem in algebraic number theory.

The author in his research efforts became interested in 2 x 2 integer matrices. Several interesting results were documented in the technical report [3]. The results reported in this research paper deal with number-theoretic concepts/ideas applied to 2 x 2 integer matrices.

This research paper is organized as follows. In Section 2, composition of binary quadratic forms arising in the case of 2 x 2 symmetric integer matrices are discussed. In Section 3, trinary quadratic forms naturally arising in association with symmetric 2x 2 integer matrices are identified and interesting results are derived. In Section 4, composition of binary quadratic forms naturally arising in association with certain structured quadratic forms arising in structured 2 x 2 matrices is discussed. The research paper concludes in Section 5.

2. 2 x 2 Symmetric Integer Matrices: Sums of Squares of Two Integers: Compositions:

Consider a symmetric 2 x 2 integer matrix of the form $X = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ $where \ \{ \ a, b, c \ \} \ are \ integers$. It readily follows that $X^2 = \begin{bmatrix} a^2+b^2 & b(a+c) \\ b(a+c) & b^2+c^2 \end{bmatrix}$. Thus, we have

$Det(X^2)$ = (Det(X))$^2 = \ (a^2+b^2)(b^2+c^2) - b^2(a+c\ )^2$.
Using the fact that Trace (X) = a + c , we have that

(Det(X))$^2 = \ (a^2+b^2)(b^2+c^2) - b^2(\ Trace(X)\ )^2$.

Hence,
(Det (X))$^2 + (\ b\ (\ Trace(X)\ ))^2 = (a^2+b^2)(b^2+c^2)$.
Using the standard identity on product of sum of squares of two integers, we have thata

$$(a^2+b^2)(b^2+c^2) = (\ ab+bc\ )^2 + (\ ac-b^2)^2 = (\ ab-bc\ )^2 + (\ ac+b^2)^2.$$

Thus, genus theorem from algebraic number theory readily applies. Let X_1, X_2 be two symmetric integer matrices with elements $\{\ a_1, b_1, c_1\ \}$; $\{\ a_2, b_2, c_2\ \}$ respectively. Using the above discussion, we have that

$$(Det(X_1))^2 + (\ b_1(Trace(X_1))^2 = \ (a_1^2+b_1^2)(b_1^2+c_1^2)$$
$$(Det(X_2))^2 + (\ b_2(Trace(X_2))^2 = \ (a_2^2+b_2^2)(b_2^2+c_2^2)$$

Since the LHS as well as RHS of the above two expressions are binary quadratic Forms, genus theorem can be readily invoked. The binary quadratic form associated with the symmetric matrix, X is based on the quantity

$$(\ \mathrm{Det}\ (X)\ )^2 + (\ b\ (\ Trace(X)\ ))^2 \ = \ (\ \mu_1\mu_2)^2 + b^2(\mu_1+\mu_2)^2, \text{ where}$$
$$\mu_1, \mu_2 \ \ are \ eigenvalues \ of \ \ X.$$

3. **2 x 2 Symmetric Integer Matrices: Sum of Squares of 3 integers (Trinary Quadratic Form): Equivalence of Trinary and Binary Quadratic Forms:**

We now consider another interesting quantity associated with a 2 x 2 symmetric integer matrix: $X = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$. We readily have that

$$X^2 = \begin{bmatrix} a^2 + b^2 & b(a+c) \\ b(a+c) & b^2 + c^2 \end{bmatrix}.$$

$Thus,\ Trace(X^2) = \ a^2 + 2\,b^2 + c^2 = \ \mu_1^2 + \mu_2^2$, where

μ_1, μ_2 *are eigenvalues of X, which are in general quadratic algebraic numbers.*

- It readily follows that all the 8 possible combinations of components of X i.e.

$\{ \mp a, \mp b, \mp c \}$ will all lead to the same value of $Trace(X^2)$.

- The eigenvalues are "complimentary" quadratic surds in the sense that if $\mu_1 = \delta + \sqrt{\theta}$, then $\mu_2 = \delta - \sqrt{\theta}$ with $\mu_1 + \mu_2 = Trace\ (X)$.

Note: We thus have a trinary quadratic form in integers equal to the binary quadratic form in eigenvalues of symmetric 2 x 2 integer matrix. Since X is symmetric, its eigenvalues are real numbers. We now determine the nature of eigenvalues under some conditions on the integers a, b, c.

CASE I : Eigenvalues are rational numbers/integers:

$\mu_1 + \mu_2 = \ a + c$ and $\mu_1 \mu_2 \ = \ ac - b^2$.

Also, $(\mu_1 - \mu_2)^2 = (a-c)^2 + \ \ (2\,b)^2 = \Delta^2$

Hence, we have that $\mu_1,\ \mu_2 \ = \ \frac{(a+c) \mp \sqrt{(a-c)^2 + \ (2\,b)^2}}{2}$. Ir readily follows that the eigenvalues depend on { a, c } through { a+c, a-c } only.

Thus, if { (a-c), (2b), Δ } form a Pythagorean triple, the eigenvalues of X are rational Numbers i.e. $\mu_1,\ \mu_2 \ = \ \frac{(a+c) \mp \ \Delta}{2}$. If { (a+c), Δ } are even integers, then the both the eigenvalues are integers.

Note: { (a+c), (a-c) } are both even/odd integers. Hence, if { a, c } are both even/odd, the eigenvalues will be integers.

CASE (ii): $(a-c)^2 + \ \ (2\,b)^2$ = p, a prime number.

By Fermat's Theorem, $p \equiv 1 (mod)\ 4$.

For any given p, (a-c)= k is unique by Fermat's Theorem.

In such case, a+c = k + 2 c. Thus, $\mu_1, \ \mu_2 \ = \ \frac{(k+2c) \mp \ \sqrt{p}}{2}$.

Hence, in this case, $\mu_1, \ \mu_2$ are real algebraic numbers.

Note: A related case is the one, where$(a-c)^2 + \ (2\,b)^2 = q$, an integer which is not a perfect square (even or odd number). Even in this case the eigenvalues are algebraic numbers.

- INTERESTING TRINARY QUADRATIC FORM:
  We now focus on the following equation from the above discussion:
  $$, \qquad Trace(X^2) = \ a^2 + 2\,b^2 + c^2 = \ \ \mu_1^2 + \mu_2^2$$

  We reduce the above equality of trinary and binary quadratic forms (with $\mu_1, \ \mu_2$ being real algebraic numbers) to the case of equality between two binary quadratic forms under some conditions:

  (I) { a, c, d } form a Pythagorean triple
  $$Trace(X^2) = \ a^2 + 2\,b^2 + c^2 = \ d^2 + 2\,b^2 = \mu_1^2 + \mu_2^2.$$

  If the eigenvalues $\{\mu_1, \ \mu_2\}$ are integers, then the genus Theorem associated With binary quadratic forms can be invoked. Also, letting X_1, X_2 be two such 2 x 2 Symmetric integer matrices, we have

$$Tr(X_1^2) = \ d_1^2 + 2b_1^2 \ \ ; \ \ Tr(X_2^2) = \ d_2^2 + 2b_2^2\,.$$

We can readily invoke Brahmagupta’s identity for the composition of such quadratic forms

- Bramhagupta’s Identity:
  For a given n, the product of two numbers of the form $a^2 + nb^2$ is itself a number of that form i.e.
  $$(a^2 + nb^2)(c^2 + nd^2) \ = \ (ac - nbd)^2 + n(ad + bc)^2$$
  $$= (ac + nbd)^2 + n(ad - bc)^2$$

  The identity holds in any commutative ring.

We now invoke the Brahmagupta’s identity:

$$Tr(X_1^2) \ Tr(X_2^2) = (\,d_1^2 + 2b_1^2\,)(d_2^2 + 2b_2^2\,) = (d_1 d_2 - 2b_1 b_2)^2 + 2\,(d_1 b_2 + d_2 b_1)^2$$
$$= (d_1 d_2 + 2b_1 b_2)^2 + 2\,(d_1 b_2 - d_2 b_1)^2$$

Note: The other interesting cases which lead to composition of binary quadratic forms are

$$a^2 + 2b^2 = e^2 \ \ or \ \ c^2 + 2b^2 = f^2, \ \ for\ suitable\ integers\ \{\,e, f\,\}.$$

- From linear algebra, we readily have that

$$Tr(X^{2m}) = \mu_1^{2m} + \mu_2^{2m} \ \ for \ m \geq 1.$$

As in the case of m=1, the above expression reduces to interesting polynomial in { a, b, c }

- **Fermat's Theorem on Sum of Squares of Two Integers: Pell's Equation: Connections:**

The following Fermat's theorem is well known.

Fermat's Theorem: Given a prime, p of the form, $p \equiv 1 \ (mod)\ 4$, it can be uniquely expressed as sum of squares of two positive integers i.e $p = a^2 + c^2 \ for \ unique \ positive \ integers \ \{ a, c \}$. Furthermore, a prime $q \equiv 3 \ (mod)\ 4$ can never be expressed as sum of squares of two positive integers.

The author observed that for some such primes, p *i.e* $p = a^2 + c^2$, *a + c is also a prime, q. But 'q'* can be such that $q \equiv 1 \ (mod)\ 4 \ or \ q \equiv 3 \ (mod)\ 4$. Some examples are

$$1^2 + 2^2 = 5 \ \ with \ 1 + 2 = 3 \ ; \ \ 2^2 + 5^2 = 29 \ \ with \ 2 + 5 = 7 \equiv 3 \ (mod)\ 4$$

Also, $4^2 + 5^2 = 41 \, with \ 4 + 5 = 9$. It can be confirmed by numerical evidence.

We interpret the observation from the viewpoint of 2 x 2 symmetric integer matrices. Let μ_1, μ_2 be the eigenvalues of symmetric 2 x 2 integer matrix, $\bar{X}$ such that

$$\mu_1 + \mu_2 = Trace(\bar{X}) = q, \qquad a \ prime \ number \ \ and$$

$\mu_1^2 + \mu_2^2 = Trace(\bar{X}^2) = p, \ a \ \ \ prime\, number.$ Also let

$$\mu_1 \mu_2 = Determinant\ (\bar{X}) = r.$$

It readily follows that $q^2 = p + \ 2 \ r.\ Suppose\ 'r'\ is\ a\ perfect\ square\ of\ an\ integer\, i.e.$

$r = b^2.\ We\ readily\ have\ a\ Diophantine\ equation\ \ q^2 - 2\ b^2 = \ p\ which\, is\ a$

$Pell-type\ \ equation.$

In view of the above discussion, we want to synthesize a 2 x 2 symmetric integer matrix,

$$X = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \ with\ a + c = q\ and\ a^2 + c^2 = p.\ If\ \bar{X}\, is\ a\ diagonal\ matrix, then\, \{a, c\}\, are$$

the eigenvalues and Fermat's theorem with the above observation applies.

Now, suppose $\bar{X}$ is not a diagonal matrix , but it is a singular matrix i.e. $a\, c = \ b^2$. We readily have that $(a + c)^2 = \ q^2 = \ a^2 + c^2 + 2\, b^2 = p + 2\, b^2$. Thus, in this case Pell-type Diophantine equation naturally follows. We now provide some examples:

$$X = \begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \ with\ a + c = 5\ ,\ a^2 + c^2 = 17\ and\ b = 2.\ In\ the\ same\ spirit, we\ have$$

$$\begin{bmatrix} 1 & 6 \\ 6 & 36 \end{bmatrix} \ and\ \begin{bmatrix} 9 & 6 \\ 6 & 4 \end{bmatrix}.\ In\ all\ the\ three\ examples, \qquad we\ have\ that$$

$$a = (\tilde{a})^2, \qquad c = (\tilde{c})^2 \;\; and \;\; b = \tilde{a}\,\tilde{c}. \; Hence \; L^1 - norm\,(\,\bar{X}\,) = a + c + 2b = (\,\tilde{a} + \tilde{c})^2.$$

Hence, in these three examples

$$a + c = (\tilde{a})^2 + (\tilde{c})^2 \; is \; a \; prime \; and \; a^2 + c^2 = (\tilde{a})^4 + (\tilde{c})^4 \; is \; also \; a \; prime.$$

The above examples motivate the theme of investigating the nature of primes, p expressible as $(\tilde{a})^4 + (\tilde{c})^4 = p$.

In view of the connection between Fermat's Theorem and Pell's equation, we introduce an interesting class of Diophantine equations:

Primal Diophantine Equation: An algebraic equation in which the variables are constrained to be primes is called a PRIMAL DIOPHANTINE EQUATION i.e. integer solutions in a Diophantine equation are constrained to be PRIMES.

We now reason that the Pell-type Diophantine equation that we derived above i.e.

$q^2 - 2\; s = \; p = q^2 - 2\; a\,c = \; p$ is never a PRIMAL Diophantine equation ($i.e.\; p \; and \; q \; and \; s \; are \; all \; never \; primes$) if the eigenvalues of the associated symmetric matrix $X = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$, $are \; integers. \; Trivially \;\; a = 1 \; and$ *c= r, a prime or a= r and c=1*. Since a+c is constrained to be prime, r = 2. It readily follows that in such case $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \; or \; \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \; will \; have \; eigenvalues \; that \; are \; complimentary \; quadratic \; surds.$

Thus, such a matrix is the only 2 x 2 symmetric integer matrix with the associated Pell-type Diophantine equation being a PRIMAL Diophantine equation.

- Now, we consider representation of a prime number, p using the specific trinary quadratic form considered above.

$$Trace(X^2) = \;\; a^2 + 2\,b^2 + c^2 = p = \mu_1^2 + \mu_2^2.$$

We provide some examples which illustrate the fact that $p \equiv 1(mod)4 \;\; or \;\; p \equiv 3\;(mod)4$. It readily follows (from Fermat's Theorem) that, if $Trace(X^2) = q, \; a \; prime \; with \; q \equiv 3\;(mod)\;4$, then, $\{\; \mu_1, \mu_2 \,\} \; are \; real \; quadratic \; algebraic \; numbers \; (\, quadratic \; surds\,) \; and \; not \; integers.$

Example 1: $\bar{X} = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$. We have that $Trace(X^2) = 13 \equiv 1\;(mod)\;4. \; Also$

$$Trace(X) = \mu_1 + \; \mu_2 \;\; = 3 \;\; and \;\; determinant\;(\,X\,) = \mu_1\mu_2 \;\; = -2.$$

$$\mu_1 = \frac{3 + \sqrt{17}}{2}, \qquad \mu_2 = \frac{3 - \sqrt{17}}{2}$$

i.e. eigenvalues are quadratic surds

Example 2: $\bar{X} = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$. We have that $Trace(X^2) = 23 \equiv 3(mod)\;4. \; Also$

$$Trace(X) = \mu_1 + \; \mu_2 \;\; = 3 \;\; and \;\; determinant\;(\,X\,) = \mu_1\mu_2 \;\; = -7.$$

$$\mu_1 = \frac{3+\sqrt{37}}{2}, \qquad \mu_2 = \frac{3-\sqrt{37}}{2}$$

Note: In both the examples, the eigenvalues are NOT integers.

- Suppose the 2 x 2 symmetric integer matrix, $\bar{X}$ is singular (i.e. $b^2 = ac$). In this case, we have that Det($\bar{X}$) $= \mu_1\mu_2$ 0. $Furthermore,$ $Trace(X^2) = a^2 + 2\,b^2 + c^2 = \mu^2$. $Note$: We can call such 4 integers { a,b,c, μ | as Pythagorean Quadruples.

Note: If $Trace(\bar{X}^2) = p, a\ prime\ number$, then $\bar{X}$ cannot be singular.

In the above two examples, we realized that if $Trace(\bar{X}^2)$ is a prime, p , it can be such that $p \equiv 1\ (mod\ )\ 4\ or\ p \equiv 3\ (mod)\ 4,\quad when\ the\ eigenvalues\ are\ real\ quadratic\ algebraic$ *numbers*. The following Lemma is in the spirit of Fermat's theorem which states that if

$$J^2 + K^2 = q,$$

then $q \equiv 1(\,mod\,)\,4$ and given 'q'; { J, K } are unique.

Lemma : Given $\bar{X}$ is a 2 x 2 symmetric integer matrix with given trace value and $Trace(X^2) = a^2 + 2\,b^2 + c^2 = \mu_1^2 + \mu_2^2 = g,\ with\ g\ being\ a\ prime\ and\ \mu_1, \mu_2$ being the eigenvalues of $\bar{X}$ that are "complimentary" quadratic surds (but not integers), { μ_1, μ_2 } are unique.

Proof: The Lemma follows from the fact that $Trace(\bar{X})$ and $Trace(X^2)$ uniquely determine the pair of eigenvalues { μ_1, μ_2 } . Q. E.D.

Note: The 2 x 2 matrices $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ and $\begin{bmatrix} c & b \\ b & a \end{bmatrix}$ have the same set of eigenvalues. Hence, given prime g, the associated 2 x 2 matrix is not unique. In fact, based on the sign, there are 8 associated 2 x 2 symmetric matrices (i.e.with $\pm a, \pm b, \pm c$) which lead to the same value of g, a prime number. It is possible, to determine the conditions under which two 2 x 2 symmetri integer matrices have the same value of $Trace(\bar{X}^2)$.

The following Theorem deals with { $Trace(\bar{X}^s)\ for\ s \geq 2$ }.

THEOREM: Let $\bar{X} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ be a 2 x 2 symmetric integer non-singular, non-diagonal matrix (i.e. $b^2 \neq ac\ \ and\ b \neq 0$) and let { a, c } are both even or both are odd. Also, let { (a-c), 2 b, K } form a Pythagorean triple i.e. $(a-c)^2 + \ (2\,b\,)^2 = k^2$. Let $Trace(\bar{X}^s) = f_s(\,a, b, c\,)\ be\ a\ trivariate\ polynomial\ in\ \{\,a, b, c\,\}$.

Under these conditions,

(i) we have that $Trace(\bar{X}^s) \neq \mu^s\ for\ any\ integer\ s \geq 2$.

(ii) Also, $Det\,(\bar{X}^s) = \beta^s = (Det(\bar{X}\,)^s = (ac - b^2)$ for all s.

(iii) If $\bar{X}$ is singular, then $Trace(\bar{X}^s) = \delta^s \; for \; all \; s$
$(where \; \delta \; is \; the \; non-zero \; eigenvalue \; of \; \bar{X}$.

PROOF: From the conditions in the statement of theorem (based on earlier discussion) that the eigenvalues of $\bar{X} \; are \; integers \; and \; Trace(\bar{X}^s) = \mu_1^s + \mu_2^s \; for \; all \; s \geq 1.$

From Fermat's Last Number Theorem, we have that

$$Trace(\bar{X}^s) \neq \mu^s \; for \; any \; integer \; s \geq 3.$$

Now, let us consider the case of s=2. It readily follows that (for $\bar{X}$ non-singular)

$$\mu_1^2 + \mu_2^2 = \frac{(a+c)^2 + k^2}{2}.$$

The RHS in the above equation cannot be an even or odd integer (based on properties of even/odd integers). Thus, the result in (i) follows.

The claims in (ii), (iii) follow from basic linear algebra results Q. E. D.

Note: From the above Lemma it follows that on the manifold/hypersurface $f_p(a,b,c) - \mu^p$, there are no integral/rational points (i.e. a,b,c,μ's) for $p \geq 2$. But for p=1, there are infinitely many rational/integral points on the associated plane (a+c-μ)

Note: $f_2(a,b,c) = a^2 + 2b^2 + c^2$, $f_3(a,b,c) = a^3 + 3ab^2 + 3b^2c + c^3$ and

$f_4(a,b,c) = (a^2+b^2)^2 + 2(b(a+c))^2 + (c^2+b^2)^2$ i.e. $f_4(a,b,c)$ is of the same form as $f_2(a,b,c)$ $i.e. \; weighted \; sum \; of \; three \; squares \; of \; integers$. Further, It can be readily reasoned that $f_m(a,b,c) \; is \; a \; sum \; of \; squares \; of \; 3 \; integers \; for \; any \; even \; 'm$.

Note: By the conditions in the above theorem, we have that the associated 4-manifolds $f_m(a,b,c) - \mu^m$ have no rational/integral points on them for $m \geq 2$.

Note: In the spirit of known theme in number theory, we are interested in 2 x 2 symmetric integer matrices for which $f_2(a,b,c) = a^2 + 2b^2 + c^2$ is a prime number. Also, we are interested in the case where : $f_m(a,b,c) = \alpha^2 + 2\beta^2 + \gamma^2 \; is \; a \; prime \; number$ $(with \; 'm' \; being \; an \; even \; number)$

- Suppose the conditions of the above Theorem are violated i.e. the eigenvalue are not integers but are "complimentary quadratic surds". Then, we provide an example where $\mu_1^p + \mu_2^p = \mu^p \; can \; happen \; for \; p = 2.$

Example: $\bar{X} = \begin{bmatrix} 3 & 10 \\ 10 & 4 \end{bmatrix}$. $The \; eigenvalues \; of \; \bar{X} \; are \; \mu_1 = \frac{7+\sqrt{401}}{2}, \; \mu_2 = \frac{7-\sqrt{401}}{2}$

$which \; are \; complimentary \; quadratic \; surds$. It follows that $\mu_1^2 + \mu_2^2 = 15^2$.

$$Thus \; f_2(a,b,c) = a^2 + 2b^2 + c^2 = f_2(a,b,c) = 3^2 + 2(10^2) + 4^2.$$

$But \; f_3(a,b,c) = 2191$ which is not the perfect cube of any integer. We are led to the following conjecture in the spirit of Fermat's Last Number Theorem based on quadratic surds.

CONJECTURE: Given that μ_1, μ_2 are "complimentary" quadratic surds which are the eigenvalues of a non-singular, 2 x 2 symmetric integer matrix. For any $p \geq 3,\ we\ have\ that$

$$\mu_1^p + \mu_2^p \neq \mu^p \ \ for\ an\ \ integer\ \ \mu.$$

We provide the following facts related to the above conjecture. Let the eigenvalues of $\bar{X}$ be the "complimentary" quadratic surds i.e. $\mu_1 = \frac{\delta+\sqrt{\gamma}}{2},\ \mu_2 = \frac{\delta-\sqrt{\gamma}}{2}$. It readily follows that

$$\mu_1^p + \mu_2^p = (\frac{\delta+\sqrt{\gamma}}{2})^p + (\frac{\delta-\sqrt{\gamma}}{2})^p \ = \mu^p = \frac{1}{2^p}\begin{bmatrix} \sum_j^p \binom{p}{j} \delta^{p-j} \sqrt{\gamma}^j \\ j = 0 \ \ and\ j\ even \end{bmatrix} = \ \mu^p \ \ for\ \ p \geq 2.$$

In the case of *p = 3, p=4 we are led to the following Diophantine equations in* δ, γ.

p = 3..............$\delta^3 - 4\mu^3 + 3\ \delta\ \gamma = 0.$

p = 4..............$\delta^4 + 6\ \delta^2\gamma + \gamma^2 - 8\mu^4\ \gamma = 0.$

The conjecture boils down to proving that the above Diophantine equations have no solutions when the 2 x 2 symmetric integer matrix is non-singular.

4-Manifolds: Reduction to Lower Dimensional Surface: Parametrization using Eigenvalues:

Let $K_p(\mu_1, \mu_2) = \mu_1^p + \mu_2^p$. We readily have that $K_p(\mu_1, \mu_2) - \theta^p = f_p(a, b, c) - \theta^p$. Thus, using the eigenvalues of $\bar{X}$ as the variables (parametrization), a 4-manifold/hyper surface is reduced to a lower dimensional surface

Note: We generalize the essential idea of above Theorem to arbitrary finite dimensional symmetric square integer matrices. Let B be a symmetric N x N dimensional integer matrix. Let the real eigenvalues of such a matrix be $\mu_i\ \ for\ \ i = 1\ \ to\ \ N$. It readily follows that

$Tr(\bar{B}^p) = \ \sum_{i=1}^N \mu_i^p$. Also $Tr(\bar{B}^p)$ is a function of $M = \frac{N(N+1)}{2}$ variables. Let

$h_p(a_1, a_2, \dots, a_M) = \mu^p$ is a manifold/hypersurface in the variables $a_1, a_2, \dots, a_M$.

Hence, based on the known results in number theory, the number of rational/integral points on such a manifolds/hypersurface can be studied. It also readily follows that

$$h_p(a_1, a_2, \dots, a_M) = \sum_{i=1}^N \mu_i^p = \ \mu^p,$$

leads to an associated manifold in the variables $\mu_i\ \ for\ \ i = 1\ \ to\ \ N$. We expect interesting connections between number theory, group theory and topology of manifolds.

Note: As specified in the case of 2 x 2 matrices, using eigenvalues of $\bar{X}$ as the variables (by parametrization), an M+1 dimensional hypersurface/manifold is reduced to an N+1 (lower dimensional) dimensional hypersurface.

Note: As stated earlier, it can readily be reasoned that $f_p(a, b, c)\ is\ always\ a\ sum\ of\ squares$ $of\ three\ integers\ if\ 'p'\ is\ an\ even\ number.\ This\ fact\ will\ compliment\ the$

$claims\ \ of\ \ the\ \ above\ \ Thoerem.$

In view of the above results, we formulate an interesting generalization of Waring problem.

GENERALIZED WARING PROBLEM:

- In the simplest case, determine the number of ways in which an integer, f (from natural numbers) can be expressed as sum of squares of two real quadratic algebraic numbers, μ_1, μ_2 (quadratic surds) i.e

$$\mu_1^2 + \mu_2^2 = f\ .$$

More generally, we are interested in number of representations of the following form:,

$$\mu_1^2 + \mu_2^2 + \cdots . + \mu_M^2 = f\ .$$

with μ_i's *being quadratic algebraic numbers*.

In the spirit of above generalization, we can consider weighted sum of squares. Most generally, we consider number of representations of an integer as a general multi-variate polynomial (could be homogeneous) in quadratic algebraic numbers. Generalizations in the spirit of Hilbert's 10th problem are possible (with the variables being quadratic surds instead of integers from the natural numbers).

4. **Structured 2 x 2 Integer Matrices: Composition of Binary Quadratic Forms:**

We now consider structured 2 x 2 integer matrices. First we consider a 2 x 2 integer matrix, X of the following form:

$$\bar{X} = \begin{bmatrix} a & -\alpha\, b \\ b & a \end{bmatrix}\ , \qquad where\ \ \alpha\ \ is\ \ an\ \ integer.$$

Such class of matrices reduce to the 2 x 2 matrices representing complex numbers when $\alpha = 1$.

It readily follows that the determinant of such a structured matrix is an interesting binary quadratic form:

$$Det(\bar{X}) = \quad a + \alpha\, b^2.$$

Given two such integer matrices $\{\bar{X}_1, \bar{X}_2\ \}$, it readily follows that

$$Det(\ \bar{X}_1 \bar{X}_2) = Det(\bar{X}_1)\ Det(\bar{X}_2\ ).$$

Thus, the RHS in the above equation can be expressed as the interesting binary quadratic form using the Bramhagupta's identity. Details are avoided for brevity.

Now, we consider a structured 2 x 2 integer matrix of the following form:

$\bar{X} = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$. It readily follows that $\bar{X}^2 = \begin{bmatrix} a^2 + b^2 & 0 \\ 0 & a^2 + b^2 \end{bmatrix} = (a^2 + b^2)\, I$.

It readily follows that compositions of binary quadratic forms can be readily invoked in association with two diagonal matrices $\bar{X}^2, \bar{Y}^2$ (associated with structured matrices $\bar{X}, \bar{Y}$).

Infact several number- theoretic results (such as MATRIX PYTHAGOREAN THEOREM, MATRIX GENUS THEOREM) can be readily invoked in association with such structured 2 x 2 integer matrices.

5. CONCLUSIONS:

In this research paper, several interesting results related to composition of binary quadratic forms arising in connection with 2 x 2 integer matrices are explored. It is expected that these results have interesting implications to algebraic number theory based on quadratic surds. Theorem in Section 3 could also lead to new insights into topology of manifolds.

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