The Continuity of Fuzzy Numbers

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April 18, 2024

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Abstract

In this paper, we studied the way below relation in the set of fuzzy numbers. It is found that there is no way below relation for interval fuzzy numbers, whereas the way below relation exists for point numbers. Additionally, the set of point fuzzy numbers as poset is continuous.

Keywords: Fuzzy number, Way below, Compact element, Continuous

1. Introduction and preliminaries

In 1965, Zadeh [11] introduced the concept of fuzzy sets. Fuzzy numbers are a kind of fuzzy sets that are of particular importance in fuzzy set theory. The research on fuzzy numbers is extensive, such as [1, 2, 3, 4, 7, 8, 9, 10]. The set of fuzzy numbers forms a poset according to the classical order[11] of fuzzy sets, and the way below relation plays a crucial role in order theory. Therefore, it is essential to investigate the way below relation in the set of fuzzy numbers.

Definition 1.1. [6] A fuzzy number is a function $u : \mathbb{R} \to [0, 1]$ satisfying the following conditions:

(1) u is normal, that is, there is a real number t_0 such that $u(t_0) = 1$;

(2) u is compactly supported, that is, the closure of $\{t \in \mathbb{R} | u(t) > 0\}$ is bounded;

(3) u is convex, that is, $r \leq t \leq s$ implies $\min\{u(r), u(s)\} \leq u(t)$ for all $r, s, t \in \mathbb{R}$;

(4) u is upper semi-continuous, that is, $\{t | u(t) \geq \alpha\}$ is closed for each $\alpha \in [0, 1]$.

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The set of all fuzzy numbers is denoted by $\mathcal{F}$. Let $u \in \mathcal{F}$, if there is only one t_0, such that $u(t_0) = 1$, then u is called a point fuzzy number. If there exist $t_1 \neq t_2$, such that $u(t_1) = u(t_2) = 1$, then u is called an interval fuzzy number. The set $\mathbb{R}$ of real numbers is canonically embedded in $\mathcal{F}$, identifying each real number a with the crisp fuzzy number $\tilde{a} : \mathbb{R} \to [0, 1]$ given by

$$\tilde{a}(t) = \begin{cases} 1, & t = a; \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, crisp fuzzy numbers are special kinds of point fuzzy numbers.

Example 1.2. Let

$$u(t) = \begin{cases} t, & 0 \leq t \leq 1; \\ 0, & \text{otherwise.} \end{cases} \quad v(t) = \begin{cases} 1, & t = 1; \\ \frac{1}{4}, & \frac{1}{2} \leq t < 1; \\ 0, & \text{otherwise.} \end{cases}$$

Then u and v are both point fuzzy numbers.

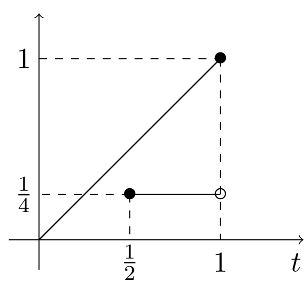

Fig. 1. u and v.

Proposition 1.3. [1] *Let $u \in \mathcal{F}, u(t_0) = 1$, then u is increasing on $(-\infty, t_0)$ and is decreasing on $(t_0, +\infty)$.*

Definition 1.4. [11] Let $u, v \in \mathcal{F}$, $u \leq v (or\ v \geq u) \Leftrightarrow \forall t \in \mathbb{R}, u(t) \leq v(t)$.

The set of fuzzy numbers $\mathcal{F}$ is a poset about $\leq$. We define $u \vee v = \min\{w | w \geq u, w \geq v\}, u \wedge v = \max\{w | w \leq u, w \leq v\}$.

Definition 1.5. [5] Let L be a preorder. A subset D of L is directed provided it is nonempty and every finite subset of D has an upper bound in D.

Definition 1.6. [5] Let L be a poset. We say that x is way below y, in symbols $x \ll y$, iff for all directed subsets $D \subseteq L$ for which $\sup D$ exists, the relation $y \leq \sup D$ always implies the existence of a $d \in D$ with $x \leq d$. An element satisfying $x \ll x$ is said to be compact.

Proposition 1.7. [5]*Let L be a poset, and $x, y \in L$, then $x \ll y \Rightarrow x \leq y$.*

Definition 1.8. [5] A poset L is called continuous if it is satisfies the axiom of approximation: $(\forall x \in L) x = \bigvee^{\uparrow} \twoheaddownarrow x$, i.e. for all $x \in L$ the set $\twoheaddownarrow x = \{u \in L | u \ll x\}$ is directed, and $x = \sup\{u \in L | u \ll x\}$.

2. The way bellow relation of fuzzy numbers

Proposition 2.1. *The crisp fuzzy number is compact.*

Proof. Let u be a crisp fuzzy number, then $u = \tilde{a}$ for some $a \in \mathbb{R}$. Let $u \leq \sup D$, then $\exists v \in D$, such that $v(a) = 1$, then $u \leq v$, thus $u \ll u$. i.e. u is compact. □

Proposition 2.2. *Let u be an interval fuzzy number, then there is no fuzzy number v, such that $v \ll u$.*

Proof. Since u is an interval fuzzy number, thus there exist $t_1 < t_2$, such that $u(t_1) = u(t_2) = 1$.

Suppose there exists a v, such that $v \ll u$. Since v is normal, thus there exists a t_0, such that $v(t_0) = 1$.

Set $t_1 < s < t_2, s \neq t_0, 0 < \varepsilon, \delta < 1$, and set

$$w_{\varepsilon,\delta}(t) = \begin{cases} \min\{u(t), 1-\varepsilon\}, & t < t_1; \\ 1-\varepsilon, & t_1 \leq t < s; \\ 1, & t = s; \\ 1-\delta, & s < t \leq t_2; \\ \min\{u(t), 1-\delta\}, & t \geq t_2. \end{cases}$$

Then the set of $w_{\varepsilon,\delta}, 0 < \varepsilon, \delta < 1$, is directed and $\sup\limits_{0<\varepsilon,\delta<1} w_{\varepsilon,\delta} = u$. Since $w_{\varepsilon,\delta}(t_0) < 1$, thus there is no $w_{\varepsilon,\delta}$ that satisfies $v \leq w_{\varepsilon,\delta}$. Hence $v \ll u$ is not true. So there is no fuzzy number v, such that $v \ll u$.

□

Proposition 2.3. *Let u be a point fuzzy number, then there exists a fuzzy number v, such that $v \ll u$.*

Proof. Since u is a point fuzzy number, thus there exists a t_0 such that $u(t_0) = 1$. Now we proceed with two cases.

Case 1: u is a crisp fuzzy number. Since $u \ll u$ by Proposition 2.1, the proof is completed.

Case 2: There exists a $t_1 \neq t_0$, such that $u(t_1) \neq 0$. Without losing generality, we suppose $t_1 < t_0$.

Set $0 < \varepsilon < u(t_1)$, and set

$$u_\varepsilon(t) = \begin{cases} 1, & t = t_0; \\ u(t_1) - \varepsilon, & t_1 \leq t < t_0; \\ 0, & \text{otherwise.} \end{cases}$$

Let $u \leq \sup D$, then $\exists v \in D$, such that $v(t_1) \geq u(t_1) - \varepsilon$. Since v is increasing on $[t_1, t_0]$, thus $u_\varepsilon \leq v$. Hence $u_\varepsilon \ll u$.

□

Theorem 2.4. *Let u be a point fuzzy number. Then the set $\twoheaddownarrow u = \{v | v \ll u\}$ is directed and $u = \bigvee^{\uparrow} \twoheaddownarrow u$.*

Proof. Let D be a directed subset, $v_1 \ll u, v_2 \ll u$, and $u \leq \sup D$, then $\exists w_1, w_2 \in D$ such that $v_1 \leq w_1, v_2 \leq w_2$. Since D is directed, thus there exists a $w_0 \in D$, such that $w_1 \leq w_0, w_2 \leq w_0$. So $v_1 \vee v_2 \leq w_0$. Hence $v_1 \vee v_2 \ll u$. Therefore $\twoheaddownarrow u_\varepsilon$ is directed.

Suppose $u(t_0) = 1, t_1 < t_0 < t_2$. Set $0 < \varepsilon < u(t_1), 0 < \delta < u(t_2)$, and set

$$v_\varepsilon(t) = \begin{cases} 1, & t = t_0; \\ u(t_1) - \varepsilon, & t_1 \leq t < t_0; \\ 0, & \text{otherwise.} \end{cases} \quad w_\delta(t) = \begin{cases} 1, & t = t_0; \\ u(t_2) - \delta, & t_0 < t \leq t_2; \\ 0, & \text{otherwise.} \end{cases}$$

Then $v_\varepsilon \ll u, w_\delta \ll u$, by Proposition 2.3. Since $\sup\limits_{t_1, t_2, \varepsilon, \delta} (v_\varepsilon, w_\delta) = u$, thus $u = \bigvee^{\uparrow} \twoheaddownarrow u$. □

Theorem 2.5. *The set of point fuzzy numbers is continuous.*

Proof. By Definition 1.8 and Theorem 2.4. □

3. Conclusion remarks

Through discussion, it is concluded that while the set of fuzzy numbers is not continuous, its subset, the set of point fuzzy numbers, is continuous. However, more research needs to be done to deepen our understanding and make further achievements.

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