Maximum Principle of Markov Regime-Switching Forward Stochastic Differential Equations with Jumps and Partial Information

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MAXIMUM PRINCIPLE OF MARKOV REGIME SWITCHING FORWARD STOCHASTIC DIFFERENTIAL EQUATIONS WITH JUMPS AND PARTIAL INFORMATION

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Abstract: : In this paper, we study a stochastic maximum principle of Markov regime switching forward stochastic differential equations with jumps and partial information. Sufficient and necessary maximum principles for optimal control under partial information are deriven.

Keywords: Stochastic maximum principle, Optimal control, Partial information, Regime switching.

1 Preliminaries

Let $B(t) = B(t, w)$, $t \geq 0$, $w \in \Omega$ and $\widetilde{\mathcal{N}}(dz, dt) = \mathcal{N}(dz, dt) - v(dz)dt$ be one-dimensional Brownian motion and an independent compensated poisson random measure, respectively, on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, P)$ satisfying the usual conditions we consider a continuous-time, finie-state, observable Markov chain $\{\alpha(t)/t \geq 0\}$. $\{\mathcal{F}\}_{t\geq 0}$ is a right-continuous, P-completed filtration to wich all of the processes defined below, including the Markov chain the Brownian motions, and the poisson random measures, are adapted. Following the convention of Elliott, Aggoun, and Moore, we identify the state space of the chain with a finite state space $\mathcal{S} = \{e_1, ..., e_D\}$,where $D \in \mathbb{N}$, $e_i \in \mathbb{R}^D$, and jthe component of e_i is the Kronecker delta δ_{ij} for each $i, j = 1, 2, ..., D$. the state space $\mathcal{S}$ is called a canonical state space and its use faciliates the mathematics.

We suppose that the chain is homogeneous and irreducible. To specify statistical or probabilistic properties of the chain α. we define the generator $\Lambda = \{\lambda_{ij}\ 1 \leq i \leq j \leq D\}$ of the chain under P. this is also called the rate matrix, or the Q-matrix. Here, for each $i, j = 1, 2, .., D$, λ_{ij} is the constant transition intensity of the chain from state e_i to state e_j at time t. Note that $\lambda_{ij} \geq 0$ for $i \neq j$ and $\sum_{j=1}^{D} \lambda_{ij} = 0$, so $\lambda_{ii} \leq 0$. In what follows for each $i, j = 1, 2, .., D$ which $i \neq j$, we suppose that $\lambda_{ij} > 0$, so $\lambda_{ii} < 0$.

Elliott, Aggoun, and Moore obtained the following semimartingale dynamics for the chain α :

$$\alpha(t) = \alpha(0) + \int_0^t \Lambda^T \alpha(u)\, du + \mathcal{M}(t)$$

where $\{\mathcal{M}(t) \setminus t \geq 0\}$ is an $\mathbb{R}^D$-valued, $\left(\{\mathcal{F}\}_{t\geq 0}, P\right)$-martingale and y^T denotes the transpose of a matrixe (or, in particular, a victor).

To model the controlled state process, we first need to introduce a set of Markov jump martingales associated with the chain α. Here we follow the results of Elliott and Elliott, Aggoun, and Moore.

For each $i, j = 1, 2, .., D$, wich $i \neq j$, and $t \in [0, \infty[$ let $J^{ij}(t)$ be the number of jumps from state e_i to state e_j up to time t. Then

$$\begin{aligned}
J^{ij}(t) &= \sum_{0\leq s\leq t} \langle \alpha(s-), e_i\rangle \langle \alpha(s), e_j\rangle \\
&= \sum_{0\leq s\leq t} \langle \alpha(s-), e_i\rangle \langle \alpha(s) - \alpha(s-), e_j\rangle \\
&= \int_0^t \langle \alpha(s-), e_i\rangle \langle d\alpha(s), e_j\rangle \\
&= \int_0^t \langle \alpha(s-), e_i\rangle \langle \Lambda^T \alpha(s), e_j\rangle ds + \int_0^t \langle \alpha(s-), e_i\rangle \langle d\mathcal{M}(s), e_j\rangle ds \\
&= \lambda_{ij} \int_0^t \langle \alpha(s-), e_i\rangle ds + m_{ij}(t),
\end{aligned}$$

where $m_{ij} = \{m_{ij}(t) \backslash t \in \tau\}$ with $m_{ij}(t) = \int_0^t \langle \alpha(s-), e_i\rangle \langle d\mathcal{M}(s), e_j\rangle$ is an $\left(\{\mathcal{F}\}_{t\geq 0}, P\right)$-martingale, the m_{ij}'s are called the basie martingales associated with the chain α.

Now, for each fined $j = 1, 2, .., D$, let $\Phi_j(t)$ be the number of jumps into state e_j up to time t.

Then

$$\begin{aligned}
\Phi_j(t) &= \sum_{i=1,\ i\neq j}^{D} J^{ij}(t) \\
&= \sum_{i=1,\ i\neq j} \lambda_{ij} \int_0^t \langle \alpha(s), e_i\rangle ds + \widetilde{\Phi}_j(t),
\end{aligned}$$

where $\widetilde{\Phi}_j(t) = \sum_{i=1,\ i\neq j}^{D} m_{ij}(t)$ and, for each $j = 1, 2, .., D$, $\widetilde{\Phi}_j(t) = \left\{\widetilde{\Phi}_j(t) \backslash t \in \tau\right\}$ is a an $\left(\{\mathcal{F}_t\}_{t\geq 0}, P\right)$-martingale.

Write for each $j = 1, 2, ..., D$

$$\lambda_j(t) = \sum_{i=1,\ i\neq j}^{D} \lambda_{ij} \int_0^t \langle \alpha(s), e_i \rangle ds \tag{1}$$

Then for each $j = 1, 2, .., D$,

$$\widetilde{\Phi}_j(t) = \Phi_j(t) - \lambda_j(t) \tag{2}$$

is an $\left(\{\mathcal{F}\}_{t\geq 0}, P\right)$-martingale.

We now introduce a Markov regime-switching Poisson random measures. Let $\mathbb{R}^+ = [0, +\infty[$ be the time index set and $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$ be a measurable space. Where $\mathcal{B}(\mathbb{R}^+)$ is the Borel σ-field generated by the open subsets of $\mathbb{R}^+$.

Let $\mathbb{R}_0 = \mathbb{R}\backslash\{0\}$ and $\mathcal{B}_0$ the Borel σ-field generated by open subset O of $\mathbb{R}_0$ whose closure $\overline{O}$ does not contain the point 0. In what follows, suppose $\mathcal{N}(dz, dt)$, is independent Poisson random measure on $(\mathbb{R}^+ \times \mathbb{R}_0, \mathcal{B}(\mathbb{R}^+) \times \mathcal{B}_0)$ under P. Assume that the Poisson random measures $\mathcal{N}(dz, dt)$ has the following compensator :

$$\eta_\alpha(dz, dt) = \nu_{\alpha(t-)}(dz\backslash t)\,\eta(dt) = \langle \alpha(t-), \nu(dz\backslash t)\rangle \eta(dt), \tag{3}$$

where $\eta(dt)$ is a σ-finite measures on $\mathbb{R}^+$ and

$$\nu(dz\backslash t) = \left(\nu_{e_1}(dz\backslash t), \nu_{e_2}(dz\backslash t), .., \nu_{e_D}(dz\backslash t)\right)^T \in \mathbb{R}^D$$

is a fuction of time t.Let us observe that for each $j = 1, 2, .., D$, $\nu_{e_j}(dz\backslash t) = \nu_j(dz\backslash t)$ is the conditional Lévy density of jump sizes of the random measures $\mathcal{N}(dz, dt)$ at time t when $\alpha(t-) = e_j$ and satisfies $\int_{\mathbb{R}_0} \min\left(1,\ z^2\right) \nu_j(dz\backslash t) < \infty$. In what follows, we shall consider only the case where $\nu(dz\backslash t)$ is a function of z, that is,

$$\nu(dz\backslash t) = \nu(dz)$$

Furthermore we assume that $\eta(dt) = dt$ and write

$$\widetilde{\mathcal{N}}_\alpha(dz, dt) := \mathcal{N}_\alpha(dz, dt) - v_\alpha(dz)\,dt, \tag{4}$$

be the compensated Markov regime-switching Poisson random measure.

We now introduce the state process $X = \{X(l)\backslash l \in [0, \infty[\}$. Suppose that we are given a set $U \subset \mathbb{R}$ and a control process $u(t) = u(t, w) : [0, \infty[\times \Omega \to U$. We also require that $\{u(t, w)\backslash t \in [0, \infty[\}$ is $\mathcal{F}_t$-predictable and has right limits. Let $X(t) = X^{(u)}(t)$ be a controlled Markov regime-switching jumps-diffusion in $\mathbb{R}$ described by the stochastic differential equation

$$\begin{aligned} dX(t) &= b(t, X(t), u(t), \alpha(t), w)\,dt + \sigma(t, X(t), u(t), \alpha(t), w)\,dB(t) \\ &\quad + \int_{\mathbb{R}_0} \eta(t, X(t), u(t), \alpha(t), z, w)\,\widetilde{\mathcal{N}_\alpha}(dz, dt) \\ &\quad + \gamma(t, X(t), u(t), \alpha(t), w)\,d\widetilde{\Phi}(t) \qquad 0 \leq t \leq \infty \\ X(0) &= x_0 \in \mathbb{R}. \end{aligned} \tag{5}$$

Here $b : [0,\infty[\times \mathbb{R} \times U \times \mathcal{S}\times\Omega \rightarrow \mathbb{R}$, $\sigma : [0,\infty[\times \mathbb{R} \times U \times \mathcal{S}\times\Omega \rightarrow \mathbb{R}$, $\eta : [0,\infty[\times \mathbb{R} \times U \times \mathcal{S} \times \mathbb{R}_0\times\Omega \rightarrow \mathbb{R}$ and $\gamma : [0,\infty[\times \mathbb{R} \times U \times \mathcal{S}\times\Omega \rightarrow \mathbb{R}$, $\widetilde{\mathcal{N}}_\alpha (dz, dt)$ is one-dimentional Markov regime-switching random measures definied by (4) $\widetilde{\Phi}(t) = \left(\widetilde{\Phi}_1, .., \widetilde{\Phi}_D\right)$ whith $\widetilde{\Phi}_j (t)$, $j = 1, 2, .., D$, defined by (2). In what follows, we consider the process $\{X (t) \backslash t \in [0,\infty[\}$ as the solution of (5) associated with the control process $\{u (t) \backslash t \in [0,\infty[\}$.

Let $\xi_t \subset \mathcal{F}_t$ be a given subfiltration, representing the information avialable to the controller at time t, $t \geq 0$.

The control process $u (t)$ assumed to be $\{\xi_t\}_{t\geq 0}$ predictable and with value in a convexe set $U \subset \mathbb{R}$. Let $\mathcal{A}_\xi$ be our family of ξ_t-predictable controls. Let $\mathcal{R}$ denote the set of functions $r : [0,\infty[\times \mathbb{R}_0 \rightarrow \mathbb{R}$ such that

$$\int_{\mathbb{R}_0} |\eta (t, x, u, e_i, z) r (t, z)| \nu_i (dz) < \infty$$

for all t, x.

Let $f : [0,\infty[\times \mathbb{R} \times U \times \mathcal{S}\times\Omega \rightarrow \mathbb{R}$ be adapted with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ an assume that

$$E\left[\int_0^\infty \left\{|f (t, X (t), u (t), \alpha (t), w)| + \left|\frac{\partial f}{\partial x} ((t, X (t), u (t), \alpha (t), w))\right|^2\right\} dt\right] < \infty$$

for all $u \in \mathcal{A}_\xi$.

Then we define

$$J (x, e_i, u) = E_{x,e_i}\left[\int_0^\infty f (t, X (t), u (t), \alpha (t), w) dt\right]$$

to be our performance functional, we study the probleme to find $u^* \in \mathcal{A}_\xi$ such that

$$J (x^*, e_i, u^*) = \underset{u\in\mathcal{A}_\xi}{Sup} J (x, e_i, u). \tag{6}$$

Let us define the Hamiltonian $H : [0,\infty[\times \mathbb{R} \times U \times \mathcal{S} \times \mathbb{R} \times \mathbb{R} \times \mathcal{R}\times\mathbb{R} \rightarrow \mathbb{R}$ by

$$\begin{aligned} H (t, x, u, e_i, p, q, r, s) &= f (t, x, u, e_i, w) + pb (t, x, u, e_i, w) + q\sigma (t, x, u, e_i, w) \\ &+ \int_{\mathbb{R}_0} \eta (t, x, u, e_i, z, w) r (t, z) \nu_i (dz) \qquad (7) \\ &+ \sum_{j=1}^{D} \gamma^j (t, x, u, e_i, w) s_j (t) \lambda_{ij}. \end{aligned}$$

For natational convenience we will in the rest paper subpress w from from the natation. The adjoint equation in the unknown $\mathcal{F}_t$-predictable processes

$(p(t), q(t), r(t,z), s(t))$ is the following backward stochastic differential equation (BSDE)

$$\begin{aligned} dp(t) &= -\frac{\partial H}{\partial x}(t, X(t), u(t), \alpha(t), p(t), q(t), r(t,.), s(t))\,dt \\ &\quad + q(t)\,dB(t) + \int_{\mathbb{R}_0} r(t,z)\,\widetilde{\mathcal{N}}_\alpha(dz,dt) + s(t)\,d\widetilde{\Phi}(t), \qquad t \geq 0. \end{aligned} \tag{8}$$

In the finite horizon case (replacing ∞ by a finite terminal time T in $J(x, e_i, u)$ above) the adjoint variable $p(t)$ would have the specified terminal value

$p(T) = 0$.

In the infinite horizon case it is natural to guess that the corresponding terminal condition would be

$\lim_{t\to\infty} p(t) = 0$.

However, this turns out to be incorrect : the terminal condition must be replaced by a limite inequality. See Theorems 1 and 4 this illustrates that the infinite horizon case requires new technique, and it cannot be deduced from the finite horizon case.

LaTeX

2 Optimal control with partial information and infinite horizon

Now, let up get back to the probleme of maximizing the performance functional

$$J(x, e_i, u) = E_{x,e_i}\left[\int_0^\infty f(t, X(t), u(t), \alpha(t))\,dt\right].$$

Where $X(t)$ is of the from (5). Our aim is to find a $u^* \in \mathcal{A}_\xi$ such that

$$J(x^*, e_i, u^*) = \underset{u\in\mathcal{A}_\xi}{Sup}\, J(x, e_i, u),$$

where $u(t)$ is our predictable control adapted to subfiltration $\xi_t \subset \mathcal{F}_t$, with value in a set $U \subset \mathbb{R}$.

Let H be the Hamiltonian defined by (7) and p the solution to the adjoint equation (8). Then we have the following maximum principle.

Theorem 1 (Sufficient Infinite Horizon Maximum Principle) *Let $u^* \in \mathcal{A}_\xi$ and let $(p^*(t), q^*(t), r^*(t,z), s^*(t))$ be an associated solution to Eq (8). Assume that for all $u \in \mathcal{A}_\xi$ the following terminal condition holds :*

$$0 \leq E\left[\overline{\lim_{t\to\infty}}\,[p^*(t)(X(t) - X^*(t))]\right] < \infty. \tag{9}$$

Moreover, assume that $H(t, x, u, e_i, p^*(t), q^*(t), r^*(t,.), s^*(t))$ is concave in x and u and

$$\begin{aligned} & E\left[H\left(t, X^*(t), u^*(t), \alpha(t), p^*(t), q^*(t), r^*(t,.), s^*(t)\right) \backslash \varepsilon_t\right] \\ = \; & \max_{u \in U} E\left[H\left(t, X^*(t), u, \alpha(t), p^*(t), q^*(t), r^*(t,.), s^*(t)\right) \backslash \varepsilon_t\right]. \end{aligned} \tag{10}$$

In addition we assume that for all $T = \infty$,

$$E\left[\int_0^\infty \left(X^*(t) - X^u(t)\right)^2 \left\{ \left(q^*\right)^2(t) + \int_{\mathbb{R}_0} \left(r^*\right)^2(t,z)\, \nu_\alpha(dz) + \sum_{j=1}^{D} \left(s_j^*\right)^2(t)\, \lambda_j(t) \right\} dt\right] < \infty, \tag{11}$$

and

$$E\left[\int_0^\infty \left(p^*\right)^2(t) \left\{ \left(\sigma(t) - \sigma^*(t)\right)^2 + \int_{\mathbb{R}_0} \left(\eta(t,z) - \eta^*(t,z)\right)^2 \nu_\alpha(dz) + \sum_{j=1}^{D} \left(\gamma^j - \gamma^{*j}\right)^2 \lambda_j(t) \right\} dt\right] < \infty \tag{12}$$

$$E\left[\left|\frac{\partial}{\partial u} H\left(t, X^*(t), u^*(t), \alpha(t), p^*(t), q^*(t), r^*(t,.), s^*(t)\right)\right|^2\right] < \infty, \tag{13}$$

and that

$$E\left[\int_0^\infty \left|H\left(t, X(t), u(t), \alpha(t), p^*(t), q^*(t), r^*(t,.), s^*(t)\right)\right|\right] < \infty \tag{14}$$

for all u.

Then we have that $u^*(t)$ is optimal.

Remark 2 *Note that, since $p(t)$ has the economic interpretation as the marginal value of the resource (alternatively the shawow price if representingon outside resource), the requirement*

$$0 \leq E\left[\overline{\lim_{t \to \infty}}\left[p^*(t)\left(X(t) - X^*(t)\right)\right]\right] < \infty,$$

has the economic interpretation that if the marginal value is positive at infinity we want to have as little resources left as possible.

Remark 3 *The requirement in the finite horizon case that $p(T) = 0$ does not translate into $\lim_{T \to \infty} p(T) = 0$ as was shown in the deterministic case in Halkin (1974).*

Proof. Let $I^\infty := E\left[\int_0^\infty \left\{f\left(t, X(t), u(t), \alpha(t)\right) - f\left(t, X^*(t), u^*(t), \alpha(t)\right)\right\} dt\right] = J(x, e_i, u) - J(x^*, e_i, u^*)$. ■

Then $I^{\infty}=I_1^{\infty}-I_2^{\infty}-I_3^{\infty}-I_4^{\infty}-I_5^{\infty}$, where

$$\begin{aligned} I_1^{\infty} \quad : \quad = E\Bigg[\int_0^{\infty} & \left(H\left(s,X\left(s\right),u\left(s\right),\alpha\left(s\right),p^*\left(s\right),q^*\left(s\right),r^*\left(s,.\right),s^*\left(s\right)\right)\right. \\ & \left.-H\left(s,X^*\left(s\right),u^*\left(s\right),\alpha\left(s\right),p^*\left(s\right),q^*\left(s\right),r^*\left(s,.\right),s^*\left(s\right)\right)\right)ds\Bigg], \end{aligned}$$

$$I_2^{\infty}:=E\left[\int_0^{\infty}p^*\left(s\right)\left(b\left(s,X\left(s\right),u\left(s\right),\alpha\left(s\right)\right)-b^*\left(s,X^*\left(s\right),u^*\left(s\right),\alpha\left(s\right)\right)\right)ds\right],$$

$$I_3^{\infty}:=E\left[\int_0^{\infty}q^*\left(s\right)\left(\sigma\left(s,X\left(s\right),u\left(s\right),\alpha\left(s\right)\right)-\sigma^*\left(s,X^*\left(s\right),u^*\left(s\right),\alpha\left(s\right)\right)\right)ds\right],$$

$$I_4^{\infty}:=E\left[\int_0^{\infty}\int_{\mathbb{R}_0}\left(\eta\left(s,X\left(s\right),u\left(s\right),\alpha\left(s\right),z\right)-\eta^*\left(s,X^*\left(s\right),u^*\left(s\right),\alpha\left(s\right),z\right)\right)r^*\left(s,z\right)\nu_{\alpha(s)}\left(dz\right)ds\right],$$

$$I_5^{\infty}:=E\left[\int_0^{\infty}\sum_{j=1}^{D}\left(\gamma^j\left(s,X\left(s\right),u\left(s\right),\alpha\left(s\right)\right)-\gamma^{*j}\left(s,X^*\left(s\right),u^*\left(s\right),\alpha\left(s\right)\right)\right)s_j^*\left(s\right)\lambda_j\left(s\right)ds\right].$$

Write

$$H_{t,X,u,\alpha,p^*,q^*,r^*,s^*}:=H\left(t,X\left(t\right),u\left(t\right),\alpha\left(t\right),p^*\left(t\right),q^*\left(t\right),r^*\left(t,.\right),s^*\left(t\right)\right),$$

and similar for other combinations. We have from concavity that

$$\begin{aligned} & H_{t,X,u,\alpha,p^*,q^*,r^*,s^*}-H_{t,X^*,u^*,\alpha,p^*,q^*,r^*,s^*} \\ \leq \quad & \frac{\partial}{\partial x}H\left(t,X^*\left(t\right),u^*\left(t\right),\alpha\left(t\right),p^*\left(t\right),q^*\left(t\right),r^*\left(t,.\right),s^*\left(t\right)\right)\left(X\left(t\right)-X^*\left(t\right)\right) \\ & +\frac{\partial}{\partial u}H\left(t,X^*\left(t\right),u^*\left(t\right),\alpha\left(t\right),p^*\left(t\right),q^*\left(t\right),r^*\left(t,.\right),s^*\left(t\right)\right)\left(u\left(t\right)-u^*\left(t\right)\right) \end{aligned} \tag{15}$$

Then we have from (10),(13) and that $u\left(t\right)$ is adapted to ξ_t,

$$\begin{aligned} 0 \quad \geq \quad & \frac{\partial}{\partial u}E\left[H_{t,X^*,u,\alpha,p^*,q^*,r^*,s^*}\diagdown\xi_t\right]_{u=u^*(t)}\left(u\left(t\right)-u^*\left(t\right)\right) \\ = \quad & \frac{\partial}{\partial u}E\left[H_{t,X^*,u^*,\alpha,p^*,q^*,r^*,s^*}\left(u\left(t\right)-u^*\left(t\right)\right)\diagdown\xi_t\right]. \end{aligned} \tag{16}$$

Combining (8), (11), (15) and (16), we get

$$
\begin{aligned}
I_1^{\infty} &\leq E\left[\int_0^{\infty} \frac{\partial}{\partial x} H_{t,X^*,u^*,\alpha,p^*,q^*,r^*,s^*}\left(X(s)-X^*(s)\right) ds\right] = E\left[\int_0^{\infty} dp^*(s)\left(X(s)-X^*(s)\right)\right] \\
:&= -J_1.
\end{aligned}
$$

From (9), (11), and Ito's formula (for simplicity let $\eta_{s,X,u,\alpha,z} := \eta(s, X(s), u(s), \alpha(s), z)$ and $\gamma^j_{s,X,u,\alpha} := \gamma^j(s, X(s), u(s), \alpha(s))$), we have that

$$
\begin{aligned}
0 \leq\ & E\left[\overline{\lim_{t\to\infty}}\left[p^*(t)\left(X(t)-X^*(t)\right)\right]\right] \\
& E\left[\int_0^{\infty} p^*(s)\left(b_{s,X,u,\alpha}-b_{s,X^*,u^*,\alpha}\right) ds\right. \\
& +\int_0^{\infty} p^*(s)\left(\sigma(s,X(s),u(s),\alpha(s))-\sigma^*(s,X^*(s),u^*(s),\alpha(s))\right) dB(s) \\
& +\int_0^{\infty}\int_{\mathbb{R}_0} p^*(s)\left(\eta(s,X(s),u(s),\alpha(s),z)-\eta^*(s,X^*(s),u^*(s),\alpha(s),z)\right)\widetilde{\mathcal{N}}_{\alpha}(ds,dz) \\
& +\int_0^{\infty} p^*(s)\left(\gamma(s,X(s),u(s),\alpha(s))-\gamma^*(s,X^*(s),u^*(s),\alpha(s))\right) d\widetilde{\Phi}(t)+\int_0^{\infty}\left(X(s)-X^*(s)\right) \\
& \times\left(-\frac{\partial}{\partial x}H^*(s,X^*(s),u^*(s),\alpha(s),p^*(s),q^*(s),r^*(s,.),s^*(s))\right) ds \\
& +\int_0^{\infty} q^*(s)\left(X(s)-X^*(s)\right) dB(s)+\int_0^{\infty}\int_{\mathbb{R}_0} r^*(s,z)\left(X(s)-X^*(s)\right)\widetilde{\mathcal{N}}_{\alpha}(ds,dz) \\
& +\int_0^{\infty} s^*(s)\left(X(s)-X^*(s)\right) d\widetilde{\Phi}(t) \\
& +\int_0^{\infty} q^*(s)\left(\sigma(s,X(s),u(s),\alpha(s))-\sigma^*(s,X^*(s),u^*(s),\alpha(s))\right) ds \\
& +\int_0^{\infty}\int_{\mathbb{R}_0} r^*(s,z)\left(\eta(s,X(s),u(s),\alpha(s),z)-\eta^*(s,X^*(s),u^*(s),\alpha(s),z)\right) v_{\alpha(s)}(dz)\, ds \\
& \left.+\int_0^{\infty}\sum_{j=1}^{D} s_j^*(s)\left(\gamma^j(s,X(s),u(s),\alpha(s))-\gamma^{*j}(s,X^*(s),u^*(s),\alpha(s))\right)\lambda_j(s)\, ds\right].
\end{aligned}
$$

From (11) and (12), we have that

$$
\begin{aligned}
& E\left[\int_0^\infty p^*(s)\left(b(s,X(s),u(s),\alpha(s))-b(s,X^*(s),u^*(s),\alpha(s))\right)ds+\int_0^\infty (X(s)-X^*(s))\right.\\
& \times\left(-\frac{\partial}{\partial x}H^*(s,X^*(s),u^*(s),\alpha(s),p^*(s),q^*(s),r^*(s,.),s^*(s))\right)ds\\
& +\int_0^\infty q^*(s)\left(\sigma(s,X(s),u(s),\alpha(s))-\sigma^*(s,X^*(s),u^*(s),\alpha(s))\right)ds\\
& +\int_0^\infty\int_{\mathbb{R}_0} r^*(s,z)\left(\eta(s,X(s),u(s),\alpha(s),z)-\eta^*(s,X^*(s),u^*(s),\alpha(s),z)\right)v_{\alpha(s)}(dz)\,ds\\
& \left.+\int_0^\infty\sum_{j=1}^{D}s_j^*(s)\left(\gamma^j(s,X(s),u(s),\alpha(s))-\gamma^{*j}(s,X^*(s),u^*(s),\alpha(s))\right)\lambda_j(s)\,ds\right]\\
= \;& I_{1,2}^\infty+J_1^\infty+I_{1,3}^\infty+I_{1,4}^\infty+I_{1,5}^\infty.
\end{aligned}
$$

Finally, combining the above we get

$$
\begin{aligned}
J(x,e_i,u)-J(x^*,e_i,u^*) &\leq I_1^\infty-I_2^\infty-I_3^\infty-I_4^\infty-I_5^\infty\\
&\leq -J_1^\infty-I_2^\infty-I_3^\infty-I_4^\infty-I_5^\infty\\
&\leq 0.
\end{aligned}
$$

This holds for all $u\in\mathcal{A}_\xi$, so the proof is complete.

2.1 Necessary Maximum Principle

To answer the question: if u^* is optimal does it satisfy

$$
\begin{aligned}
& E\left[H\left(t,\ X^*(t),\ u^*(t),\ \alpha(t),\ p^*(t),\ q^*(t),\ r^*(t,.),\ s^*(t)\right)/\xi_t\right]\\
&=\max_{u\in U}E\left[H\left(t,\ X^*(t),\ u,\ \alpha(t),\ p^*(t),\ q^*(t),\ r^*(t,.),\ s^*(t)\right)/\xi_t\right]
\end{aligned}
\tag{17}
$$

we assume the following:

A1 For all t_0, h such that $0\leq t_0\leq t_0+h\leq\infty$ and all bounded ξ_{t_0}-measurable random variables α, the control process $\beta(t)$ defined by

$$\beta(t)=\alpha 1_{[t_0,t_0+h]}(t),$$

belongs to $\mathcal{A}_\xi$. Here

$$1_{[t_0,t_0+h]}(t)=\begin{cases}1 & \text{if } t\in[t_0,\ t_0+h]\\ 0 & \text{otherwise.}\end{cases}$$

A2 For all $u \in \mathcal{A}_\xi$ and all $\beta \in \mathcal{A}_\xi$ bounded, there exists $\epsilon > 0$ such that

$$u + \epsilon\beta \in \mathcal{A}_\varepsilon \text{ for all } \epsilon \in [-\delta, \delta] .$$

A3 The derivative process

$$\varepsilon(t) := \left. \frac{d}{d\epsilon} X^{u+\epsilon\beta}(t) \right|_{\epsilon=0}$$

exists and belongs to $L^2(m \times P)$, where m denotes the Lebesgue measure on $\mathbb{R}$.

$$\begin{aligned}
d\varepsilon(t) &= \left\{ \frac{\partial b}{\partial x}(t)\,\varepsilon(t) + \frac{\partial b}{\partial u}(t)\,\beta(t) \right\} dt + \left\{ \frac{\partial \sigma}{\partial x}(t)\,\varepsilon(t) + \frac{\partial \sigma}{\partial u}(t)\,\beta(t) \right\} dB(t) \\
&\quad + \int_{\mathbb{R}_0} \left\{ \frac{\partial \eta}{\partial x}(t,z)\,\varepsilon(t) + \frac{\partial \eta}{\partial u}(t,z)\,\beta(t) \right\} \widetilde{\mathcal{N}}_\alpha(dt, dz) \\
&\quad + \left\{ \frac{\partial \gamma}{\partial x}(t)\,\varepsilon(t) + \frac{\partial \gamma}{\partial u}(t)\,\beta(t) \right\} d\widetilde{\Phi}(t),
\end{aligned}$$

where, for simplicity of notation, we define

$$\frac{\partial b}{\partial x}(t) := \frac{\partial b}{\partial x}(t,\ X(t),\ \alpha(t),\ u(t)) .$$

Note that

$$\varepsilon(t) = 0.$$

A4 Assume that f satisfies a Lipschitz condition of the form

$$|f(x_1, u_1, \alpha_1) - f(x_2, u_2, \alpha_2)| \leq C(t)\left(|x_1 - x_2| + |u_1 - u_2| + |\alpha_1 - \alpha_2|\right),$$

for any $t, x_i, u_i, \alpha_i,\ i = 1, 2.$

We can then give an answer to the question.

Theorem 4 (Partial Information Necessary Maximum Principle) .
Suppose $u^* \in \mathcal{A}_\xi$ is a local maximum for $J(u)$ meaning that for all bounded $\beta \in \mathcal{A}_\xi$ there exists a $\delta > 0$ such that $u^* + \epsilon\beta \in \mathcal{A}_\xi$ for all $\epsilon \in (-\delta, \delta)$ and
$h(\epsilon) := J(u^* + \epsilon\beta),\ \epsilon \in (-\delta, \delta)$
is maximal at $\epsilon = 0$. Let $(p^*(t),\ q^*(t),\ r^*(t,z),\ s^*(t))$ be the solution to the (linear) adjoint equation

$$\begin{aligned}
dp^*(t) &= -\frac{\partial H}{\partial x}(t,\ X^*(t),\ u^*(t),\ \alpha(t),\ p^*(t),\ q^*(t),\ r^*(t,.),\ s^*(t))\,dt \\
&\quad + q^*(t)\,dB(t) + \int_{\mathbb{R}_0} r^*(z,t)\,\widetilde{\mathcal{N}}_\alpha(dz, dt) + s^*(t)\,d\widetilde{\Phi}(t) .
\end{aligned}$$

Moreover assume that if $\varepsilon^*(t) = \varepsilon^{(u^*,\beta)}(t)$, *with corresponding coefficients* π_t^*, τ_t^*, $\varsigma_{t,z}^*$, φ_t^*,

where

$$\pi_t^2 = \left(\frac{\partial b_{t,X,u,\alpha}}{\partial x}\right)^2 \varepsilon^2(t) + \left(\frac{\partial b_{t,X,u,\alpha}}{\partial u}\right)^2 \beta^2(t)$$
$$\tau_t^2 = \left(\frac{\partial \sigma_{t,X,u,\alpha}}{\partial x}\right)^2 \varepsilon^2(t) + \left(\frac{\partial \sigma_{t,X,u,\alpha}}{\partial u}\right)^2 \beta^2(t)$$
$$\varsigma_{t,z}^2 = \left(\frac{\partial \eta_{t,X,u,z,\alpha}}{\partial x}\right)^2 \varepsilon(t) + \left(\frac{\partial \eta_{t,X,u,z,\alpha}}{\partial u}\right)^2 \beta^2(t)$$
$$\varphi_t^2 = \left(\frac{\partial \gamma_{t,X,u,\alpha}}{\partial x}\right)^2 \varepsilon^2(t) + \left(\frac{\partial \gamma_{t,X,u,\alpha}}{\partial u}\right)^2 \beta^2(t)$$

$$\lim_{t\to\infty} E\left[p^*(t)\,\varepsilon^*(t)\right] = 0, \tag{18}$$

$$E\left[\int_0^\infty C(t)\left(1+|\varepsilon^*(t)|\right)dt\right] < \infty, \tag{19}$$

$$E\left[\int_0^\infty (\varepsilon^*(t))^2 \left\{(q^*)^2(t) + \int_{\mathbb{R}_0} (r^*(t,z))^2 v_\alpha(dz) + \sum_{j=1}^{D} \left(\gamma^j\right)^2(t)\,\lambda_j(t)\right\} dt\right] < \infty \tag{20}$$

where $\lambda(t) = (\lambda_1(t), .., \lambda_D(t))^T$; $v_\alpha(dz) = \left(v_{\alpha(t-)}(dz), .., v_{\alpha(t-)}(dz)\right)^T$,
and

$$\left[\int_0^\infty (p^*(t))^2 \left[(\tau^*)^2(t,\ X^*(t),\ \alpha(t),\ u^*(t)) + \int_{\mathbb{R}_0} (\varsigma^*)^2(t,\ X^*(t),\ \alpha(t),\ u^*(t),\ z)\,v_\alpha(dz)\right.\right.$$
$$\left.\left. + \sum_{j=1}^{D} \left(\varphi^{j*}\right)^2(t,\ X^*(t),\ \alpha(t),\ u^*(t))\,\lambda_j(t)\right] dt\right] < \infty, \tag{21}$$

for all $T < \infty$. *Then* u^* *is a stationary point for* $E[H \setminus \xi]$ *in the sense that for all* $t \geq 0$,

$$E\left[\frac{\partial}{\partial u} H(t,\ X^*(t),\ e_i,\ u^*,\ p^*(t),\ q^*(t),\ r^*(t,.),\ s^*(t)) \setminus \xi_t\right] = 0. \tag{22}$$

Proof. For simplicity we consider only the 1-dimentional case. First note that by A3, A4 and (19) we have that

$$
\begin{aligned}
0 &= \left.\frac{\partial}{\partial \epsilon} J\left(u^{*}+\epsilon \beta\right)\right|_{\epsilon=0} \\
&= \left.\frac{\partial}{\partial \epsilon} E\left[\int_{0}^{\infty} f\left(t,\ X^{u^{*}+\epsilon \beta}(t),\ u^{*}(t)+\epsilon \beta,\ \alpha(t)\right) dt\right]\right|_{\epsilon=0} \\
&= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} E\left[\int_{0}^{\infty}\left\{f\left(t,\ X^{u^{*}+\epsilon \beta}(t),\ u^{*}(t)+\epsilon \beta,\ \alpha(t)\right)-f\left(t,\ X^{u^{*}}(t),\ u^{*}(t)\ , \alpha(t)\right)\right\} dt\right] \\
&= E\left[\int_{0}^{\infty}\left\{\frac{\partial f}{\partial x}\left(t,\ X^{u^{*}}(t),\ u^{*}(t)\ , \alpha(t)\right) \varepsilon^{*}(t)+\frac{\partial f}{\partial u}\left(t,\ X^{u^{*}}(t),\ u^{*}(t)\ , \alpha(t)\right) \beta(t)\right\} dt\right].
\end{aligned}
\tag{23}
$$

We Know by the definition of H that

$$
\frac{\partial f}{\partial x}(t)=\frac{\partial H}{\partial x}(t)-\frac{\partial b}{\partial x}(t) p(t)-\frac{\partial \sigma}{\partial x}(t) q(t)-\int_{\mathbb{R}_{0}} \frac{\partial \eta}{\partial x}(t, z) r(t, z) v_{\alpha}(dz)-\sum_{j=1}^{D} \frac{\partial \gamma^{j}}{\partial x}(t) s_{j}(t) \lambda_{j}(t)
\tag{24}
$$

and the same for $\frac{\partial f}{\partial u}(t)$.

Applying the Itô to $p^{*}(t) \varepsilon^{*}(t)$, we obtain by (18), A2, (20) and (21)

$$
\begin{aligned}
0 = & \lim_{t \rightarrow \infty} E\left[\overset{*}{p}(t) \overset{*}{\varepsilon}(t)\right] \\
& E\left[\int_{0}^{\infty} p^{*}(t)\left\{\frac{\partial b}{\partial x}(t) \varepsilon^{*}(t)+\frac{\partial b}{\partial u}(t) \beta(t)\right\} dt+\int_{0}^{\infty} \varepsilon^{*}(t)\left(-\frac{\partial H^{*}}{\partial x}\right) dt\right. \\
& +\int_{0}^{\infty} q^{*}(t)\left\{\frac{\partial \sigma}{\partial x}(t) \varepsilon^{*}(t)+\frac{\partial \sigma}{\partial u}(t) \beta(t)\right\} dt \\
& +\int_{0}^{\infty} \int_{\mathbb{R}_{0}} r^{*}(t, z)\left\{\frac{\partial \eta}{\partial x}(t, z) \varepsilon^{*}(t)+\frac{\partial \eta}{\partial u}(t, z) \beta(t)\right\} v_{\alpha}(dz) dt \\
& \left.+\int_{0}^{\infty} \sum_{j=1}^{D} s_{j}^{*}(t)\left\{\frac{\partial \gamma^{j}}{\partial x}(t) \varepsilon^{*}(t)++\frac{\partial \gamma^{j}}{\partial u}(t) \beta(t) \lambda_{j}(t)\right\} dt\right]
\end{aligned}
$$

$$
\begin{aligned}
&= E\left[\int_0^\infty \varepsilon^*(t)\left\{\frac{\partial b}{\partial x}(t)\,p^*(t)+\frac{\partial \sigma}{\partial x}(t)\,q^*(t)+\int_{\mathbb{R}_0}\frac{\partial \eta}{\partial x}(t,z)\,r^*(t,z)\,v(dz)\right.\right.\\
&\left.+\sum_{j=1}^{D}s_j(t)\,\frac{\partial \gamma^j}{\partial x}(t)\,\lambda_j(t)-\frac{\partial H^*}{\partial x}\right\}dt+\int_0^\infty \beta(t)\left\{\frac{\partial b}{\partial u}(t)\,p^*(t)+\frac{\partial \sigma}{\partial u}(t)\,q^*(t)\right.\\
&\left.\left.\int_{\mathbb{R}_0}\frac{\partial \eta}{\partial x}(t,z)\,r^*(t,z)\,v(dz)+\sum_{j=1}^{D}s_j(t)\,\frac{\partial \gamma^j}{\partial x}(t)\,\lambda_j(t)\right\}dt\right]
\end{aligned}
$$

$$
\begin{aligned}
&= E\left[\int_0^\infty \varepsilon^*(t)\left(-\frac{\partial f}{\partial x}(t)\right)dt+\int_0^\infty \beta(t)\left\{\frac{\partial H^*}{\partial u}(t)-\frac{\partial f}{\partial u}(t)\right\}dt\right]\\
&= -E\left[\int_0^\infty\left\{\frac{\partial f}{\partial x}(t)\,\varepsilon^*(t)+\frac{\partial f}{\partial u}(t)\,\beta(t)\right\}dt\right]+E\left[\int_0^\infty\frac{\partial H^*}{\partial u}(t)\,\beta(t)\,dt\right]\\
&= -\frac{d}{d\epsilon}J\left(u^*+\epsilon\beta\right)\big|_{\epsilon=0}+E\left[\int_0^\infty\frac{\partial H^*}{\partial u}(t)\,\beta(t)\,dt\right]
\end{aligned}
$$

Therefore

$$E\left(\int_0^\infty\frac{\partial H^*}{\partial u}(t)\,\beta(t)\,dt\right)=\frac{d}{d\epsilon}J\left(u^*+\epsilon\beta\right)\big|_{\epsilon=0}$$

Now apply this to

$$\beta(t)=\alpha 1_{[t_0,t_0+h]}(t),$$

where α is bounded and ξ_{t_0}-measurable, $0\le t_0\le t_0+h\le\infty$. Then if (23) holds we get

$$E\left(\int_{t_0}^{t_0+h}\frac{\partial}{\partial u}H^*(t,\ X_t^*,\ e_i,\ u_t^*,\ p_t^*,\ q_t^*,\ r^*(t,.)\,,\ s_t^*)\,dt.\alpha\right)=0.$$

Differentiating with respect to h at $h=0$, we have

$$E\left(\frac{\partial}{\partial u}H^*\left(t_0,\ X_{t_0}^*,\ e_i,\ u_{t_0}^*,\ p_{t_0}^*,\ q_{t_0}^*,\ r^*(t_0,.)\,,\ s_{t_0}^*\right)\alpha\right)=0.$$

This holds for all ξ_{t_0}-measurable α and hence we obtain that

$$E\left(\frac{\partial}{\partial u}H^*(t_0)\ \setminus\ \xi_{t_0}\right)=0.$$

Which proves the theorem. ■

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