Finite Hilbert-Style Axiomatizations of Disjunctive and Implicative Finitely-Valued Logics with Equality Determinant

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ABSTRACT
Here, we develop a unversal method of [effective] constructing a [finite] Hilbert-style axiomatization of the logic of a given finite disjunctive/implicative matrix with equality determinant [and finitely many connectives] (in particular, any/ implicative four-valued expansion of Belnap's four-valued logic /[as well as any Łukasiewicz finitely-valued logic]).

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Logic; calculus; sequent; matrix.

1. Introduction

Though various universal approaches to (mainly, *many-place*) *sequent* axiomatizations of finitely-valued logics (cf., *e.g.*, Pynko (2014) as well as both its and its references' bibliographies) have being extensively developed, the problem of their standard (viz., Hilbert-style) axiomatizations (especially, on a generic level) has deserved much less emphasis.

On the other hand, the general study Pynko (2004) has suggested a universal method of [effective] constructing a multi-conclusion *two-side* (as opposed to the above approaches) sequent calculus with structural rules and Cut Elimination Property for a given finite matrix with equality determinant [and finitely many connectives]. In this paper, providing the matrix involved is disjunctive/implicative, we advance the mentioned study by [effective] transforming any [finite] *sequential table for* the matrix (viz., skeletons of introduction rules for the matrix and all compound non-nullary connectives not belonging to the equality determinant) and minimal sequent axioms with disjoint sides consisting of solely either elements of the equality determinant or their values on nullary connectives true in the matrix, actually giving a Gentzen-style axiomatization of the logic of the matrix in Pynko (2004), to a [finite] Hilbert-style axiomatization of the logic.

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Our general approach, first of all, covers, aside from respective fragments of the classical logic, two especially representative infinite classes of finitely-valued logics: both four-valued expansions of Belnap's useful four-valued logic Belnap (1977), which were started to be studied in Pynko (1999) on an advanced level, and Łukasiewicz finitely-valued logics Łukasiewicz (1920). In addition, it covers miscellaneous three-valued para-consistent/-complete logics. Although most interesting of these are axiomatic/disjunctive extensions of appropriate four-valued expansions of Belnap's four-valued logic, there are certain interesting exceptions (like HZ Hałkowska and Zajac (1988)) deserving a particular emphasis, for which *finite* Hilbert-style axiomatizations have not been found yet.

The rest of the paper is as follows. We entirely follow the standard conventions (as for Hilbert-style calculi) as well as those adopted in both Pynko (1999) and Pynko (2004) — as to sequent calculi. Section 2 is a concise summary of mainly those basic issues underlying the paper, which have proved beyond the scopes of the mentioned papers, those presented therein being normally (though not entirely) briefly summarized as well for the exposition to be properly self-contained. In Section 3 we present a uniform formalism for covering both Hilbert- and Gentzen-style calculi without repeating practically same issues concerning calculi of both kinds, and recall some key results concerning disjunctive and implicative logics (mainly belonging to a logical folklore) and sequent calculi with structural rules going back to Pynko (1999). Then, Section 4 is a preliminary study of minimal disjunctive Hilbert- as well as Gentzen-style (both multi- and single-conclusion) calculi to be used further. Section 5 then contains the main generic results of the paper. Finally, in Section 6 we apply it to disjunctive and implicative positive fragments of the classical logic (with improving Dyrda and Prucnal (1980)), to Łukasiewicz finitely-valued logics and to both four-valued expansions of Belnap's four-valued logic and their three-valued extensions as well as to the three-valued logic HZ Hałkowska and Zajac (1988), applications to which prove to be especially acute, because of the infiniteness of its Hilbert-style axiomatization originally found in Zbrzezny (1990). Finally, Section 7 is a brief summary of principal *definitive* contributions of the paper.

2. Basic issues

Notations like img, dom, ker, hom, π_i, R^{-1} and $Q \circ R$ as well as related notions are supposed to be clear.

2.1. Set-theoretical background

We follow the standard set-theoretical convention, according to which natural numbers (including 0) are treated as finite ordinals (viz., sets of lesser natural numbers), the ordinal of all them being denoted by ω (cf., e.g. Mendelson (1979)). The proper class of all ordinals is denoted by ∞. Likewise, functions are viewed as binary relations. In addition, singletons are often identified with their unique elements, unless any confusion is possible.

Given a set S, the set of all subsets of S [of cardinality $\in K \subseteq \infty$] is denoted by $\wp_{[K]}(S)$. Next, any S-tuple (viz., a function with domain S) is often written in the sequence form $\bar{t}$, its s-th component (viz., the value under argument s) $\pi_s(\bar{t})$, where $s \in S$, being written as t_s, in that case. As usual, given two more sets A and B, any relation between them is identified with the equally-denoted relation between A^S and

B^S defined point-wise. Further, elements of $S^* \triangleq (S^0 \cup S^+)$, where $S^+ \triangleq (\bigcup_{i\in(\omega\setminus 1)} S^i)$, are identified with ordinary finite tuples/[comma separated] sequences [in which case, as usual, semicolon instead of comma is sometimes used as sets elements separator to avoid any confusion], the binary concatenation operation on S^* being denoted by $*$, as usual. Then, any $\diamond : (S \times S) \to S$ determines the equally-denoted mapping $\diamond : S^+ \to S$ as follows: by induction on the length (viz., domain) l of any $\bar{a} \in S^+$, put:

$$(\diamond\bar{a}) \triangleq \begin{cases} a_0 & \text{if } l = 1, \\ (\diamond(\bar{a}\restriction(l-1))) \diamond a_{l-1} & \text{otherwise.} \end{cases}$$

Likewise, given a one more set T, any $\diamond : (S \times T) \to T$ determines the equally-denoted mapping $\diamond : (S^* \times T) \to T$ as follows: by induction on the length (viz., domain) l of any $\bar{a} \in S^*$, for all $b \in T$, put:

$$(\bar{a} \diamond b) \triangleq \begin{cases} b & \text{if } l = 0, \\ a_0 \diamond (((\bar{a}\restriction(l \setminus 1)) \circ ((+1)\restriction(l-1))) \diamond b) & \text{otherwise.} \end{cases}$$

Given any $R \subseteq S^2$, put $R^1 \triangleq R$ and $R^0 \triangleq \Delta_S \triangleq \{\langle s, s\rangle \mid s \in S\}$, functions of the latter kind being said to be *diagonal*.

Let A be a set. A $U \subseteq \wp(A)$ is said to be *upward-directed*, provided, for every $S \in \wp_\omega(U)$, there is some $T \in U$ such that $(\bigcup S) \subseteq T$. An *operator over* A is any unary operation O on $\wp(A)$. This is said to be *(monotonic) [idempotent]* $\{$*transitive*$\}$ $\langle$*inductive/finitary/compact*$\rangle$, provided, for all $(B,)D \in \wp(A)$ $\langle$resp., any upward-directed $U \subseteq \wp(A)\rangle$, it holds that $(O(B))[D]\{O(O(D)\} \subseteq O(D)\langle O(\bigcup U) \subseteq \bigcup O[U]\rangle$. A *closure operator over* A is any monotonic idempotent transitive operator over A.

2.1.1. Disjunctivity versus multiplicativity

Fix any set A and any $\delta : A^2 \to A$. Given any $X, Y \subseteq A$, set $\delta(X, Y) \triangleq \delta[X \times Y]$. Then, a closure operator C over A is said to be *[K-]δ-multiplicative* [where $K \subseteq \infty$] provided

$$\delta(C(X \cup Y), a) \subseteq C(X \cup \delta(Y, a)), \tag{2.1}$$

for all $(X \cup \{a\}) \subseteq A$ and all $Y \in \wp_{[K]}(A)$.[1] Next, C is said to be *δ-disjunctive*, provided, for all $a, b \in A$ and every $X \subseteq A$, it holds that

$$C(X \cup \{\delta(a, b)\}) = (C(X \cup \{a\}) \cap C(X \cup \{b\})), \tag{2.2}$$

in which case the following clearly hold, by (2.2) with $X = \varnothing$:

$$\begin{aligned} \delta(a,b) &\in C(a), & (2.3)\\ \delta(a,b) &\in C(b), & (2.4)\\ a &\in C(\delta(a,a)), & (2.5)\\ \delta(b,a) &\in C(\delta(a,b)), & (2.6)\\ C(\delta(\delta(a,b),c)) &= C(\delta(a,\delta(b,c))), & (2.7) \end{aligned}$$

[1]In this connection, "finitely-/singularly-" means "ω-/$\{1\}$-", respectively.

for all $a, b, c \in A$.

Lemma 2.1. *Let C be a [finitary] closure operator over A. Then,* (i)$\Leftrightarrow$ (ii)$\Leftrightarrow$(iii)$\Leftarrow$ [$\Leftrightarrow$](iv)*, where:*

(i) *C is δ-disjunctive;*
(ii) (2.3)*,* (2.5) *and* (2.6) *hold and C is singularly-δ-multiplicative;*
(iii) (2.3)*,* (2.5) *and* (2.6) *hold and C is finitely-δ-multiplicative;*
(iv) (2.3)*,* (2.5) *and* (2.6) *hold and C is δ-multiplicative.*

Proof. First, (ii/iii) is a particular case of (iii/iv), respectively. [Next, (iii)$\Rightarrow$(iv) is by C's being finitary.]

Further, assume (i) holds. Consider any $(X \cup \{a, b\}) \subseteq A$ and any $c \in C(X \cup \{b\})$, in which case $\delta(c, a) \in C(X \cup \{b\})$, by (2.3). Moreover, by (2.4), we also have $\delta(c, a) \in C(X \cup \{a\})$. Thus, by (2.2), we get $\delta(c, a) \in (C(X \cup \{b\}) \cap C(X \cup \{a\}) = C(X \cup \{\delta(b, a)\})$. In this way, (ii) holds.

Next, assume (ii) holds. In that case, both (2.3) and so, by (2.6), (2.4) hold, and so does the inclusion from left to right in (2.2). Conversely, consider any $c \in (C(X \cup \{b\}) \cap C(X \cup \{a\}))$, where $(X \cup \{a, b\}) \subseteq A$. Then, by (2.6) and (2.1) with $Y = \{a\}$ and b instead of a, we have $\delta(b, c) \in C(X \cup \{\delta(a, b)\})$. Likewise, by (2.5) and (2.1) with $Y = \{b\}$ and c instead of a, we have $c \in C(X \cup \{\delta(b, c)\})$. Therefore, we eventually get $c \in C(X \cup \{\delta(a, b)\})$. Thus, (i) holds.

Finally, assume (i) holds. By induction on any $n \in \omega$, let us show that C is n-δ-multiplicative. For consider any $(X \cup \{a\}) \subseteq A$, any $Y \in \wp_n(A)$, in which case $n \neq 0$, and any $b \in C(X \cup Y)$. In case $Y = \varnothing$, (2.1) is by (2.3). Otherwise, take any $c \in Y$, in which case $Y' \triangleq (Y \setminus \{c\}) \in \wp_{n-1}(A)$, and put $X' \triangleq (X \cup \{c\}) \subseteq A$, in which case $(X' \cup Y') = (X \cup Y)$, and so $b \in C(X' \cup Y')$. Hence, by induction hypothesis, we get $\delta(b, a) \in C(X' \cup \delta(Y', a)) = C(\{c\} \cup (X \cup \delta(Y', a)))$. Moreover, by (2.4), we have $\delta(b, a) \in C(\{a\} \cup (X \cup \delta(Y', a)))$. Therefore, as $Y = (Y' \cup \{c\})$, by (2.2), we eventually get $\delta(b, a) \in C(\{\delta(c, a)\} \cup (X \cup \delta(Y', a))) = C(X \cup \delta(Y, a))$. Thus, as $(\bigcup \omega) = \omega$, we conclude that C is finitely-δ-multiplicative, and so (iii) holds, as required. □

2.2. Algebraic background

Unless otherwise specified, all along the paper, we deal with a fixed but arbitrary signature Σ of *primary (propositional) connectives* of finite arity to be treated as operation (viz., function) symbols. Given any $\alpha \in \wp_{\infty \setminus 1}(\omega)$, $\mathfrak{Fm}^{\alpha}_{\Sigma}$ denotes the absolutely-free Σ-algebra freely-generated by the set $V_\alpha \triangleq \{x_i \mid i \in \alpha\}$ of *(propositional) variables*, its endomorphisms/elements of its carrier $\mathrm{Fm}^{\alpha}_{\Sigma}$ being called *(propositional) Σ-substitutions/formulas*, in case $\alpha = \omega$. As usual, a *secondary (propositional) connective of Σ of arity* $n \in \omega$ is any element of $\mathrm{Fm}^{\max(n,1)}_{\Sigma}$, any primary $F \in \Sigma$ of arity $n \in \omega$ being naturally identified with the secondary one $F(x_i)_{i \in n}$. The finite set of all variables *actually* occurring in a $\varphi \in \mathrm{Fm}^{\omega}_{\Sigma}$ is denoted by $\mathrm{Var}(\varphi)$. For any $\Pi \subseteq \mathrm{Fm}^{\omega}_{\Sigma}$, set $\mathrm{Fm}^{\alpha}_{\Pi} \triangleq (\bigcap \{V_\alpha \subseteq S \subseteq \mathrm{Fm}^{\alpha}_{\Sigma} \mid \forall \sigma \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{Fm}^{\omega}_{\Sigma}) : (\sigma[V_\omega] \subseteq S) \Rightarrow (\sigma[\Pi] \subseteq S)\}) \subseteq \mathrm{Fm}^{\alpha}_{\Sigma}$.

As usual, *(logical) Σ-matrices* (cf. Łoś and Suszko (1958)) are treated as first-order model structures of the first-order signature $\Sigma \cup \{D\}$ with unary *truth* predicate D. In general, [Σ-matrices are denoted by Calligraphic letters (possibly, with indices), their *underlying*] algebras [viz., their Σ-reducts] being denoted by [corresponding] Fraktur letters (possibly, with [same] indices [if any]), their carriers being denoted by corresponding Italic letters (with same indices, if any) [any Σ-matrix $\mathcal{A}$ being traditionally identified with the couple $\langle\mathfrak{A}, D^{\mathcal{A}}\rangle$].

2.2.1. Equality determinants for matrices

According to Pynko (2004), an *equality determinant for* a Σ-matrix $\mathcal{A}$ is any $\Upsilon \subseteq \mathrm{Fm}^1_\Sigma$ such that any $a, b \in A$ are equal, whenever, for all $\upsilon \in \Upsilon$, $\upsilon^{\mathfrak{A}}(a) \in D^{\mathcal{A}}$ iff $\upsilon^{\mathfrak{A}}(b) \in D^{\mathcal{A}}$.

3. Abstract propositional languages and calculi

A(n) *(abstract)* Σ-*[propositional]language* is any triple of the form $L = \langle \mathrm{Fm}_L, \Im_L, \mathrm{Var}_L\rangle$, where Fm_L is a set, whose elements are called *L-formulas*, while $\Im_L : \mathrm{hom}(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{Fm}^\omega_\Sigma) \to (\mathrm{Fm}_L)^{\mathrm{Fm}_L}$, preserving compositions and diagonality, any Σ-substitution σ being naturally identified with $\Im_L(\sigma)$, unless any confusion is possible, whereas $\mathrm{Var}_L : \mathrm{Fm}_L \to \wp_\omega(V_\omega)$, the language subscript being normally omitted, unless any confusion is possible, such that, for every $\Phi \in \mathrm{Fm}_L$ and any Σ-substitutions σ and ς such that $(\sigma \restriction \mathrm{Var}_L(\Phi)) = (\varsigma \restriction \mathrm{Var}_L(\Phi))$, it holds that $\sigma(\Phi) = \varsigma(\Phi)$.

Then, elements/subsets of $\mathrm{Ru}_L \triangleq (\wp_\omega(\mathrm{Fm}_L) \times \mathrm{Fm}_L)$ are referred to as *L-rules/calculi*, any L-rule $\mathcal{R} = \langle\Gamma, \Phi\rangle$ being normally written in either conventional displayed $\frac{\Gamma}{\Phi}$ or non-displayed $\Gamma|\Phi$ form, Φ/any element of Γ being called the/a *conclusion/premise of* $\mathcal{R}$, rules of the form $\Phi|\Psi$, where $\Psi \in \Gamma$, being said to be *inverse to* $\mathcal{R}$. As usual, L-rules without premises are called *L-axioms* and are identified with their conclusions, calculi consisting of merely axioms being said to be *axiomatic*. In general, any function f with domain Fm_L (including Σ-substitutions) but Var_L determines the equally-denoted function with domain Ru_L as follows: for any $\mathcal{R} = \langle\Gamma, \Phi\rangle \in \mathrm{Ru}_L$, we set $f(\mathcal{R}) \triangleq \langle f[\Gamma], f(\Phi)\rangle$, whereas put $\mathrm{Var}_L(\mathcal{R}) \triangleq (\mathrm{Var}_L(\Phi) \cup \bigcup \mathrm{Var}_L[\Gamma]) \in \wp_\omega(V_\omega)$.

Next, an *L-logic* is any closure operator C on Fm_L that is *structural* in the sense that, for every Σ-substitution σ and all $\Gamma \subseteq \mathrm{Fm}_L$, it holds that $\sigma[C(\Gamma)] \subseteq C(\sigma[\Gamma])$. This is said to *satisfy* an L-rule $\Gamma|\Phi$, whenever $\Phi \in C(\Gamma)$. Then, an L-logic C' is said to be an *extension of* C, provided $C \subseteq C'$. In that case, an L-calculus $\mathcal{C}$ is said to *axiomatize* C' *relatively to* C, provided C' is the least extension of C satisfying each rule in $\mathcal{C}$.

Further, an L-rule $\Gamma|\Phi$ is said to be *derivable in* an L-calculus $\mathcal{C}$, if there is a $\mathcal{C}$-*derivation of* it (viz., a $\mathcal{C}$-*derivation of* Φ *from* Γ), i.e., a proof of Φ (in the standard proof-theoretical sense) by means of axioms in Γ (as *hypotheses*) and rules in the set $\mathrm{SI}_\Sigma(\mathcal{C}) \triangleq \{\sigma(\mathcal{R}) \mid \mathcal{R} \in \mathcal{C}, \sigma \in \mathrm{hom}(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{Fm}^\omega_\Sigma)\}$ of all *substitutional* Σ-*instances of* rules in $\mathcal{C}$. The extension $\mathrm{Cn}_\mathcal{C}$ of the diagonal Σ-logic relatively axiomatized by $\mathcal{C}$ is called the *consequence of* $\mathcal{C}$ and said to be *axiomatized by* $\mathcal{C}$, in which case it is finitary and satisfies any L-rule iff this is derivable in $\mathcal{C}$. (Conversely, any finitary L-logic is axiomatized by the set of all L-rules satisfied in it to be identified with the logic, in which case finitary L-logics become actually particular cases of L-calculi.) An $S \subseteq \mathrm{Fm}^\omega_\Sigma$ is said to be $\mathcal{C}$-*closed*, if, for every $(\Gamma|\Phi) \in \mathrm{SI}_\Sigma(\mathcal{C})$, it holds that $(\Gamma \subseteq S) \Rightarrow (\Phi \in S)$, in which case, by induction on the length of $\mathcal{C}$-derivations, it is $\mathrm{Cn}_\mathcal{C}$-closed, that is, $S \in (\mathrm{img}\, \mathrm{Cn}_\mathcal{C})$, and so, in particular, $\mathrm{Cn}_\mathcal{C}(\varnothing) \subseteq S$.

3.1. Hilbert-style calculi

The Σ-language $\mathcal{H}_\Sigma$ with first component $\mathrm{Fm}^\omega_\Sigma$, the diagonal second component and the third component Var is called the *Hilbert-style/sentential* Σ-*language*, $\mathcal{H}_\Sigma$-rules/-

axioms/-calculi/-logics being traditionally referred to as *(Hilbert-style/sentential)* Σ-*rules/-axioms/-calculi/-logics*, respectively (cf., e.g., Loś and Suszko (1958)).

From the model-theoretic point of view, any Σ-rule $\Gamma|\phi$ is viewed as the first-order basic Horn formula $(\bigwedge\Gamma) \to \phi$ under the standard identification of any Σ-formula ψ with the first-order atomic formula $D(\psi)$ we follow tacitly.

Given any class M of Σ-matrices, we have the Σ-logic Cn_{M} *of/defined by* it, defined by $\mathrm{Cn}_{\mathsf{M}}(X) \triangleq (\mathrm{Fm}^{\omega}_{\Sigma} \cap \bigcap\{h^{-1}[D^{\mathcal{A}}] \supseteq X | \mathcal{A} \in \mathsf{M}, h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})\})$, for all $X \subseteq \mathrm{Fm}^{\omega}_{\Sigma}$. (Due to Loś and Suszko (1958), this is well known to be finitary, whenever both M and all members of it are finite.)

A Σ-matrix $\mathcal{A}$ is said to be $\diamond$-*disjunctive/-implicative*, where $\diamond$ is a (possibly, secondary) binary connective of Σ, whenever, for all $a, b \in A$, it holds that $((a \notin / \in D^{\mathcal{A}}) \Rightarrow (b \in D^{\mathcal{A}})) \Leftrightarrow ((a \diamond^{\mathfrak{A}} b) \in D^{\mathcal{A}})/$, in which case it is $\veebar_{\diamond}$-disjunctive, where $(x_0 \veebar_{\diamond} x_1) \triangleq ((x_0 \diamond x_1) \diamond x_1)$.

3.1.1. Disjunctive sentential logics

Throughout the rest of the paper, unless otherwise specified, $\veebar$ is supposed to be any (possibly, secondary) binary connective of Σ.

Lemma 3.1. *Let* M *be a class of* $\veebar$*-disjunctive* Σ*-matrices. Then, the logic of* M *is* $\veebar$*-multiplicative, and so* $\veebar$*-disjunctive.*

Proof. Consider any $(X \cup Y \cup \{\psi\}) \subseteq \mathrm{Fm}^{\omega}_{\Sigma}$, any $\phi \in \mathrm{Cn}_{\mathsf{M}}(X \cup Y)$, any $\mathcal{A} \in \mathsf{M}$ and any $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$ such that $(h(\phi) \veebar^{\mathfrak{A}} h(\psi)) = h(\phi \veebar \psi) \notin D^{\mathcal{A}}$, in which case $h(\phi) \notin D^{\mathcal{A}} \not\ni h(\psi)$, for $\mathcal{A}$ is $\veebar$-disjunctive, and so $h(\varphi) \notin D^{\mathcal{A}}$, for some $\varphi \in (X \cup Y)$, in which case $h(\varphi \veebar \psi) = (h(\phi) \veebar^{\mathfrak{A}} h(\psi)) \notin D^{\mathcal{A}}$, and so $(\phi \veebar \psi) \in \mathrm{Cn}_{\mathsf{M}}(X \cup (Y \veebar \psi))$. Then, Lemma 2.1(iv)⇒(i) completes the proof, for Cn_{M} satisfies (2.3), (2.5) and (2.6). □

Given a Σ-rule $\Gamma|\phi$ and a Σ-formula ψ, put $((\Gamma|\phi) \veebar \psi) \triangleq ((\Gamma \veebar \psi)|(\phi \veebar \psi))$. (This notation is naturally extended to Σ-calculi member-wise.)

Theorem 3.2. *Let* C *be a finitary* Σ*-logic. Then,* C *is* $\veebar$*-disjunctive iff* (2.3), (2.5) *and* (2.6) *hold and, for any axiomatization* $\mathcal{C}$ *of* C*, every* $(\Gamma|\phi) \in \mathrm{SI}_{\Sigma}(\mathcal{C})$ *and each* $\psi \in \mathrm{Fm}^{\omega}_{\Sigma}$*, it holds that* $(\phi \veebar \psi) \in C(\Gamma \veebar \psi)$*.*

Proof. By Corollary 2.1(i)⇔(iv) and the structurality of C, with using (2.3) and the induction on the length of $\mathcal{C}$-derivations. □

Lemma 3.3. *Let* $\mathcal{R} = (\Gamma|\phi)$ *be a* Σ*-rule,* C *a* Σ*-logic,* $\psi \in \mathrm{Fm}^{\omega}_{\Sigma}$*,* $\sigma \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{Fm}^{\omega}_{\Sigma})$ *and* $v \in (V_{\omega} \setminus \mathrm{Var}(\mathcal{R}))$*. Suppose* (2.7) *holds and* $\mathcal{R} \veebar v$ *is satisfied in* C*. Then, so is* $\sigma(\mathcal{R} \veebar v) \veebar \psi$*.*

Proof. Let $\varsigma \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{Fm}^{\omega}_{\Sigma})$ extend $(\sigma \restriction (V_{\omega} \setminus \{v\})) \cup [v/(\sigma(v) \veebar \psi)]$, in which case $\sigma(\mathcal{R}) = \varsigma(\mathcal{R})$, for $v \notin \mathrm{Var}(\mathcal{R})$. Then, using (2.7) and the structurality of C, we eventually get $(\sigma(\phi \veebar v) \veebar \psi) = ((\sigma(\phi) \veebar \sigma(v)) \veebar \psi) \in C(\sigma(\phi) \veebar (\sigma(v) \veebar \psi)) = C(\varsigma(\phi) \veebar \varsigma(v)) = C(\varsigma(\phi \veebar v)) \subseteq C(\varsigma[\Gamma \veebar v]) = C(\varsigma[\Gamma] \veebar \varsigma(v)) = C(\sigma[\Gamma] \veebar (\sigma(v) \veebar \psi)) = C((\sigma[\Gamma] \veebar \sigma(v)) \veebar \psi) = C(\sigma[\Gamma \veebar v] \veebar \psi)$, as required. □

Let σ_{+1} be the Σ-substitution extending $[x_i/x_{i+1}]_{i \in \omega}$.

Corollary 3.4. *Let* C *be a finitary* $\veebar$*-disjunctive logic,* $\mathcal{C}$ *a* Σ*-calculus and* $\mathcal{A}$ *an axiomatic* Σ*-calculus. Then, the extension* C' *of* C *relatively axiomatized by* $\mathcal{C}' \triangleq (\mathcal{A} \cup (\sigma_{+1}[\mathcal{C}] \veebar x_0))$ *is* $\veebar$*-disjunctive.*

Proof. Then, C being finitary, is axiomatized by a Σ-calculus $\mathcal{C}''$, in which case C' is axiomatized by the Σ-calculus $\mathcal{C}'' \cup \mathcal{C}'$, and so is finitary too. Moreover, C', being an extension of C, inherits (2.3), (2.5), (2.6) and (2.7) held for C. Then, we prove the $\veebar$-disjunctivity of C' with applying Theorem 3.2 to both C and C'. For consider any Σ-substitution σ and any $\psi \in \mathrm{Fm}_\Sigma^\omega$. First, for any $\phi \in \mathcal{A} \subseteq \mathcal{C}'$, by the structurality of C' and (2.3), we have $(\sigma(\phi) \veebar \psi) \in C'(\varnothing)$. Now, consider any $\mathcal{R} \in \mathcal{C}$, in which case $(\sigma_{+1}(\mathcal{R}) \veebar x_0) \in \mathcal{C}'$ is satisfied in C' and $x_0 \in (V_\omega \setminus \mathrm{Var}(\sigma_{+1}(\mathcal{R})))$. In this way, Lemma 3.3 with C' and $\sigma_{+1}(\mathcal{R})$ instead of C and $\mathcal{R}$, respectively, completes the argument. $\square$

3.1.2. Implicative sentential logics

Throughout the rest of the paper, unless otherwise specified, $\rhd$ is supposed to be any (possibly, secondary) binary connective of Σ.

A Σ-logic C is said to be $\rhd$*-implicative*, whenever it has *Deduction Theorem* (*DT*, for short) *with respect to* $\rhd$ in the sense that:

$$(\psi \in C(\Gamma \cup \{\phi\})) \Rightarrow ((\phi \rhd \psi) \in C(\Gamma), \tag{3.1}$$

for all $(\Gamma \cup \{\phi, \psi\}) \subseteq \mathrm{Fm}_\Sigma^\omega$, as well as satisfies both the *Modus Ponens* rule:

$$\frac{x_0 \quad x_0 \rhd x_1}{x_1}, \tag{3.2}$$

and *Peirce Law* axiom (cf. Peirce (1885)):

$$(((x_0 \rhd x_1) \rhd x_0) \rhd x_0). \tag{3.3}$$

As it is well-known, C satisfies the following axioms:

$$x_0 \rhd (x_1 \rhd x_0) \tag{3.4}$$
$$(x_0 \rhd (x_1 \rhd x_2)) \rhd ((x_0 \rhd x_1) \rhd (x_0 \rhd x_2)) \tag{3.5}$$

whenever it has DT with respect to $\rhd$ and satisfies (3.2).

Lemma 3.5. *Any $\rhd$-implicative Σ-logic is $\veebar_\rhd$-disjunctive.*

Proof. With using Lemma 2.1(ii)⇒(i). First, (2.3) is by (3.2) and (3.1). Next, (2.5) is by (3.2) and (3.3)$[x_1/x_0]$. Further, by (3.1), (3.2) and (3.3), we have $x_0 \in C(\{x_0 \veebar_\rhd x_1, x_1 \rhd x_0\})$, in which case, by (3.1), we get $(x_1 \veebar_\rhd x_0) \in C(x_0 \veebar_\rhd x_1)$, and so (2.6) holds. Finally, consider any $(\Gamma \cup \{\phi, \psi\}) \subseteq \mathrm{Fm}_\Sigma^\omega$ and any $\varphi \in C(\Gamma \cup \{\phi\})$, in which case, by (3.1), we have $(\phi \rhd \varphi) \in C(\Gamma)$, and so, by (3.2) and (3.5), we get $\psi \in C(\Gamma \cup \{\phi \veebar_\rhd \psi, \varphi \rhd \psi\})$. Hence, by (3.1), we eventually get $(\varphi \veebar_\rhd \psi) \in C(\Gamma \cup \{\phi \veebar_\rhd \psi\})$. Thus, C is singularly-$\veebar_\rhd$-multiplicative, as required. $\square$

By $\mathcal{I}_\rhd^{[\mathrm{PL}]}$ we denote the Σ-calculus constituted by (3.2), (3.4) and (3.5) [as well as (3.3)]. Recall the following well-known observation proved by induction on the length of $(\mathcal{I}_\rhd \cup \mathcal{A})$-derivations (cf., e.g., Mendelson (1979)):

Lemma 3.6. *Let $\mathcal{A}$ be an axiomatic Σ-calculus. Then, $\mathrm{Cn}_{\mathcal{I}_\rhd \cup \mathcal{A}}$ has DT with respect to $\rhd$.*

Combining Lemmas 3.5 and 3.6, we eventually get:

Theorem 3.7. *Let $\mathcal{A}$ be an axiomatic Σ-calculus. Then, $\mathrm{Cn}_{\mathcal{I}^{\mathrm{PL}}_{\rhd}\cup\mathcal{A}}$ is $\rhd$-implicative, and so $\veebar_{\rhd}$-disjunctive.*

Corollary 3.8. *Let $\mathcal{A}\cup\{\varphi\}$ be an axiomatic Σ-calculus, $n \in (\omega \setminus 1)$, $\bar{\psi} \in (\mathrm{Fm}^{\omega}_{\Sigma})^n$, $\bar{\phi} \in (\mathrm{Fm}^{\omega}_{\Sigma})^*$, $v \in (V_\omega \setminus (\bigcup \mathrm{Var}[\{\varphi\} \cup ((\mathrm{img}\,\bar{\psi}) \cup (\mathrm{img}\,\bar{\phi}))]))$ and $\bar{\zeta} \triangleq \langle \bar{\phi} \rhd (\psi_i \rhd v)\rangle_{i\in n}$. Then, the following hold:*

(i) *the Σ-axiom $\bar{\phi}\rhd((\veebar_{\rhd}\bar{\psi})\rhd\varphi)$ is derivable in $\mathcal{I}^{\mathrm{PL}}_{\rhd}\cup\mathcal{A}$ iff the Σ-axioms $\bar{\phi}\rhd(\psi_i\rhd\varphi)$, where $i \in n$, are so;*
(ii) *the Σ-axiom $\bar{\phi} \rhd (\varphi \rhd (\veebar_{\rhd}\bar{\psi}))$ is derivable in $\mathcal{I}^{\mathrm{PL}}_{\rhd} \cup \mathcal{A}$ iff the Σ-axiom $(\bar{\zeta} \rhd (\bar{\phi} \rhd (\varphi \rhd v))$ is so.*

Proof. In that case, by Theorem 3.7, $\mathrm{Cn}_{\mathcal{I}^{\mathrm{PL}}_{\rhd}\cup\mathcal{A}}$ is $\rhd$-implicative and $\veebar_{\rhd}$-disjunctive. Then, (2.2) with $X = (\mathrm{img}\,\bar{\phi})$, (3.1), (3.2) and the induction on n immediately yield (i). Next, the "if" part of (i) with v and $\bar{\zeta}*\bar{\phi}$ instead of φ and $\bar{\phi}$, respectively, (3.1) and (3.2) yield the "only if" part of (ii). Conversely, applying the substitution $[v/(\veebar_{\rhd}\bar{\psi})]$, the "only if" part of (i) with $\veebar_{\rhd}\bar{\psi}$ instead of φ, (3.2) and (3.4) imply the "if" part of (ii), as required. □

3.2. Gentzen-style calculi

Given any $(\alpha[\cup\beta]) \subseteq \omega$, elements of $\mathrm{Seq}^{[\beta\vdash]\alpha}_{\Sigma} \triangleq \{\langle\Gamma,\Delta\rangle \in ((\mathrm{Fm}^{\omega}_{\Sigma})^*)^2 \mid (\mathrm{dom}\,\Delta) \in \alpha\,[\&\,(\mathrm{dom}\,\Gamma) \in \beta]\}$ are called *α-conclusion [β-premise] Σ-sequents*, "[purely] single/multi" standing for "$(2/\omega)[\setminus 1]$", respectively. Any sequent $\langle\Gamma,\Delta\rangle$ is normally written in the conventional form $\Gamma \vdash \Delta$. This is said to be *injective*, whenever both Γ and Δ are so. Likewise, it is said to be *disjoint*, whenever $((\mathrm{img}\,\Gamma) \cap (\mathrm{img}\,\Delta)) = \varnothing$. For any $\Phi = (\Gamma \vdash \Delta) \in \mathrm{Seq}^{[\beta\vdash]\alpha}_{\Sigma}$, set $\mathrm{Var}(\Phi) \triangleq (\bigcup \mathrm{Var}[\mathrm{img}(\Gamma * \Delta)]) \in \wp_\omega(V_\omega)$ and $\sigma(\Phi) \triangleq ((\sigma\circ\Gamma) \vdash (\sigma\circ\Delta)) \in \mathrm{Seq}^{[\beta\vdash]\alpha}_{\Sigma}$, where σ is a Σ-substitution. In this way, $\mathrm{Seq}^{[\beta\vdash]\alpha}_{\Sigma}$ forms a Σ-language $\mathcal{S}^{[\beta\vdash]\alpha}_{\Sigma}$, called the *$\alpha$-conclusion [$\beta$-premise] Gentzen-style/sequent Σ-language*, $\mathcal{S}^{[\beta\vdash]\alpha}_{\Sigma}$-rules/-axioms/-calculi/logics being referred to as *α-conclusion [β-premise] (Gentzen-style/sequent) Σ-rules/-axioms/-calculi/-logics*, respectively.

The following multi-conclusion sequent $\varnothing$-rules are said to be *structural*:

Reflexivity	$x_0 \vdash x_0$	
Cut	$\dfrac{\Lambda,\Gamma\vdash\Delta,x_0 \quad \Gamma,x_0\vdash\Delta,\Theta}{\Lambda,\Gamma\vdash\Delta,\Theta}$	
Enlargement	$\dfrac{\Gamma\vdash\Delta}{x_0,\Gamma\vdash\Delta}$	$\dfrac{\Gamma\vdash\Delta}{\Gamma\vdash\Delta,x_0}$
Contraction	$\dfrac{x_0,x_0,\Gamma\vdash\Delta}{x_0,\Gamma\vdash\Delta}$	$\dfrac{\Gamma\vdash\Delta,x_0,x_0}{\Gamma\vdash\Delta,x_0}$
Permutation	$\dfrac{\Lambda,x_0,x_1,\Gamma\vdash\Delta}{\Lambda,x_1,x_0,\Gamma\vdash\Delta}$	$\dfrac{\Gamma\vdash\Delta,x_0,x_1,\Theta}{\Gamma\vdash\Delta,x_1,x_0,\Theta}$

where $\Lambda,\Gamma,\Delta,\Theta \in V^*_\omega$, Enlargement, Contraction and Permutation being referred to as *basic structural.*

Given two (purely) multi-conclusion [{purely} multi-premise] Σ-sequents $\Phi = (\Gamma \vdash \Delta)$ and $\Psi = (\Lambda \vdash \Theta)$, we have their *sequent subsumption/disjunction/implication*:

$$(\Phi \sqsubseteq \Psi) \overset{\mathrm{def}}{\Longleftrightarrow} (((\mathrm{img}\,\Gamma) \subseteq (\mathrm{img}\,\Lambda))\&((\mathrm{img}\,\Delta) \subseteq (\mathrm{img}\,\Theta)))/$$
$$(\Phi \uplus \Psi) \triangleq (\Gamma,\Lambda \vdash \Delta,\Theta) \in \mathrm{Seq}^{[(\omega\{\setminus 1\})\vdash](\omega(\setminus 1))}_{\Sigma}\,/$$

$$\begin{aligned}(\Phi \sqsupset \Psi) &\triangleq \{\phi, \Gamma \vdash \Delta \mid \phi \in (\mathrm{img}\,\Theta)\} \\ &\cup \{\Gamma \vdash \Delta, \psi \mid \psi \in (\mathrm{img}\,\Lambda)\} \in \wp_\omega(\mathrm{Seq}_\Sigma^{[(\omega\{\backslash 1\})\vdash](\omega(\backslash 1))}),\end{aligned}$$

respectively. Then, given any $X \in \wp_{\langle\omega\rangle}(\mathrm{Seq}_\Sigma^{[(\omega\{\backslash 1\})\vdash](\omega(\backslash 1))})$, set $(\Phi \sqsupset X) \triangleq (\bigcup\{\Phi \sqsupset \Psi \mid \Psi \in X\} \in \wp_{\langle\omega\rangle}(\mathrm{Seq}_\Sigma^{[(\omega\{\backslash 1\})\vdash](\omega(\backslash 1))})$.

A (purely) multi-conclusion [{purely} multi-premise] sequent Σ-calculus $\mathcal{G}$ is said to be ⟨*deductively*⟩ *multiplicative*, provided, for every (purely) multi-conclusion [{purely} multi-premise] sequent Σ-rule $X|\Phi$ ⟨derivable⟩ in $\mathcal{G}$ and each multi-conclusion Σ-sequent Ψ, the rule $(X \uplus \Psi)|(\Phi \uplus \Psi)$ is derivable in $\mathcal{G}$. Using induction on the length of $\mathcal{G}$-derivations, it is routine checking that $\mathcal{G}$ is multiplicative iff it is deductively so.

Theorem 3.9 (cf. the proof of Theorem 4.2 of Pynko (1999)**).** *Let $\mathcal{G}$ be a ⟨multiplicative⟩ (purely) multi-conclusion [{purely} multi-premise] sequent Σ-calculus with basic structural rules and Cut⟨/Reflexivity⟩ and $(X \cup \{\Phi, \Psi\}) \subseteq \mathrm{Seq}_\Sigma^{[(\omega\{\backslash 1\})\vdash](\omega(\backslash 1))}$. Then,*

$$\Psi \in \mathrm{Cn}_{\mathcal{G}}(X \cup \{\Phi\}) \Leftarrow \langle/\Rightarrow\rangle (\Phi \sqsupset \Psi) \subseteq \mathrm{Cn}_{\mathcal{G}}(X).$$

From the model-theoretic point of view, any Σ-sequent $\Gamma \vdash \Delta$ is treated as the first-order basic clause (viz., disjunct) $\bigvee(\neg[\mathrm{img}\,\Gamma] \cup (\mathrm{img}\,\Delta))$ of the signature $\Sigma \cup \{D\}$ under the notorious identification of any Σ-formula φ with the first-order atomic formula $D(\varphi)$, any sequent Σ-rule being interpreted as the universal closure of the implication of its premises (under the natural identification of any finite set X of first-order formulas with $\bigwedge X$ we follow tacitly as well) and its conclusion, in which case sequent Σ-calculi become universal first-order theories. (In this way, sequent disjunction/implication corresponds to the usual disjunction/implication.) This fits the standard matrix interpretation of sequents equally adopted in Pynko (1999) and Pynko (2004).

4. Basic disjunctive calculi

4.1. The Hilbert-style calculus

By $\mathcal{D}_{\veebar}$ we denote the Σ-calculus constituted by the following Σ-rules:

$$\begin{array}{cccc} D_1 & D_2 & D_3 & D_4 \\ \dfrac{x_0 \veebar x_0}{x_0} & \dfrac{x_0}{x_0 \veebar x_1} & \dfrac{(x_0 \veebar x_1) \veebar x_2}{(x_1 \veebar x_0) \veebar x_2} & \dfrac{(x_0 \veebar (x_1 \veebar x_2)) \veebar x_3}{((x_0 \veebar x_1) \veebar x_2) \veebar x_3} \end{array}$$

Lemma 4.1. *Let $\mathcal{C} \supseteq \mathcal{D}_{\veebar}$ be a Σ-calculus, $\mathcal{R} = (\Gamma|\phi)$ a Σ-rule and $v \in (V_\omega \setminus \mathrm{Var}(\mathcal{R}))$. Suppose $\mathcal{R} \veebar v$ is derivable in $\mathcal{C}$. Then, so is $\mathcal{R}$ itself.*

Proof. First, for every $\psi \in \Gamma$, by $D_2[x_0/\psi, x_1/\phi]$, we have $(\psi \veebar \phi) \in \mathrm{Cn}_{\mathcal{C}}(\psi)$, and so we get $(\Gamma \veebar \phi) \in \mathrm{Cn}_{\mathcal{C}}(\Gamma)$. Then, applying $(\mathcal{R} \veebar v)[v/\phi]$, by the structurality of $\mathrm{Cn}_{\mathcal{C}}$, we conclude that $(\phi \veebar \phi) \in \mathrm{Cn}_{\mathcal{C}}(\Gamma)$. Finally, $D_1[x_0/\phi]$ completes the argument. □

Applying Lemma 4.1 to both D_3 and D_4, we immediately get:

Corollary 4.2. *The following rules are derivable in* $\mathcal{D}_{\veebar}$*:*

$$\frac{x_0 \veebar x_1}{x_1 \veebar x_0}, \tag{4.1}$$

$$\frac{x_0 \veebar (x_1 \veebar x_2)}{(x_0 \veebar x_1) \veebar x_2}. \tag{4.2}$$

Lemma 4.3. *The following rules are derivable in* $\mathcal{D}_{\veebar}$*:*

$$\frac{(x_0 \veebar x_1) \veebar x_2}{x_0 \veebar (x_1 \veebar x_2)}, \tag{4.3}$$

$$\frac{(x_0 \veebar x_0) \veebar x_1}{x_0 \veebar x_1}, \tag{4.4}$$

$$\frac{x_0 \veebar x_2}{(x_0 \veebar x_1) \veebar x_2}. \tag{4.5}$$

Proof. First, in view of Corollary 4.2, (4.3) is by the following $\mathrm{Cn}_{\mathcal{D}_{\veebar}}$-derivation:

(1) $(x_0 \veebar x_1) \veebar x_2$ — hypothesis;
(2) $(x_1 \veebar x_0) \veebar x_2$ — D_3: 1;
(3) $x_2 \veebar (x_1 \veebar x_0)$ — (4.1)$[x_0/(x_1 \veebar x_0), x_1/x_2]$: 2;
(4) $(x_2 \veebar x_1) \veebar x_0$ — (4.2)$[x_0/x_2, x_2/x_0]$: 3;
(5) $(x_1 \veebar x_2) \veebar x_0$ — $D_3[x_0/x_2, x_2/x_0]$: 4;
(6) $x_0 \veebar (x_1 \veebar x_2)$ — (4.1)$[x_0/(x_1 \veebar x_0), x_1/x_0]$: 5.

Then, in view of Corollary 4.2, (4.4) is by the following $\mathrm{Cn}_{\mathcal{D}_{\veebar}}$-derivation:

(1) $(x_0 \veebar x_0) \veebar x_1$ — hypothesis;
(2) $x_0 \veebar (x_0 \veebar x_1)$ — (4.3)$[x_1/x_0, x_2/x_1]$: 1;
(3) $(x_0 \veebar x_1) \veebar x_0$ — (4.1)$[x_1/(x_0 \veebar x_1)]$: 2;
(4) $((x_0 \veebar x_1) \veebar x_0) \veebar x_1$ — $D_2[x_0/((x_0 \veebar x_1) \veebar x_0)]$: 3;
(5) $(x_0 \veebar x_1) \veebar (x_0 \veebar x_1)$ — (4.3)$[x_0/(x_0 \veebar x_1), x_1/x_0, x_1/x_2]$: 4;
(6) $(x_0 \veebar x_1)$ — $D_1[x_0/(x_0 \veebar x_1)]$: 5.

Finally, in view of Corollary 4.2, (4.5) is by the following $\mathrm{Cn}_{\mathcal{D}_{\veebar}}$-derivation:

(1) $x_0 \veebar x_2$ — hypothesis;
(2) $(x_0 \veebar x_2) \veebar x_1$ — $D_2[x_0/(x_0 \veebar x_2)]$: 1;
(3) $x_0 \veebar (x_2 \veebar x_1)$ — (4.3)$[x_1/x_2, x_2/x_1]$: 2;
(4) $(x_2 \veebar x_1) \veebar x_0$ — (4.1)$[x_1/(x_2 \veebar x_1)]$: 3;
(5) $x_2 \veebar (x_1 \veebar x_0)$ — (4.3)$[x_0/x_2, x_2/x_0]$: 4;
(6) $(x_1 \veebar x_0) \veebar x_2$ — (4.1)$[x_0/x_2, x_1/(x_1 \veebar x_0)]$: 5;
(7) $(x_0 \veebar x_1) \veebar x_2$ — $D_3[x_0/x_1, x_1/x_0]$: 6. □

Theorem 4.4. $\mathrm{Cn}_{\mathcal{D}_{\veebar}}$ *is* $\veebar$*-disjunctive.*

Proof. With using Theorem 3.2. First, by D_1, D_2, Corollary 4.2 and Lemma 4.3(4.3), (2.3), (2.5), (2.6) and (2.7) hold for $C \triangleq \mathrm{Cn}_{\mathcal{D}_{\veebar}}$.

Next, consider any $\sigma \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{Fm}^{\omega}_{\Sigma})$, any $\psi \in \mathrm{Fm}^{\omega}_{\Sigma}$ and any $i \in (5 \setminus 1)$. The case, when $i \notin 3$, is due to Lemma 3.3 with $\mathcal{R}$ such that $D_i = (\mathcal{R} \veebar x_{i-1})$. Otherwise, we have $\mathrm{Var}(D_i) = V_i \not\ni x_i$. Then, by Lemma 4.3(4.4)/(4.5), $D_i \veebar x_i$ is derivable in $\mathcal{D}_{\veebar}$. Let $\varsigma \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{Fm}^{\omega}_{\Sigma})$ extend $(\sigma \restriction V_{\omega \setminus \{i\}}) \cup [x_i/\psi]$, in which case $\varsigma(D_i) = \sigma(D_i)$, and so, by the structurality of C, we eventually conclude that $(\sigma(D_i) \veebar \psi) = (\varsigma(D_i) \veebar \varsigma(x_i)) = \varsigma(D_i \veebar x_i)$ is derivable in $\mathcal{D}_{\veebar}$, as required. □

The following auxiliary observation has proved quite useful for reducing the number of rules of calculi to be constructed in Section 6 according to the universal method to be elaborated in Section 5:

Corollary 4.5. *Let $\phi, \psi, \varphi \in \mathrm{Fm}^{\omega}_{\Sigma}$, $v \in (V_\omega \setminus (\bigcup \mathrm{Var}[\{\phi, \psi, \varphi\}]))$ and $\mathcal{C} \supseteq \mathcal{D}_{\veebar}$ a Σ-calculus. Then, the rules $(\phi \veebar v)|(\varphi \vee v)$ and $(\psi \veebar v)|(\varphi \vee v)$ are both derivable in $\mathcal{C}$ iff the rule $((\phi \veebar \psi) \vee v)|(\varphi \vee v)$ is so.*

Proof. First of all, by Theorem 4.4, $C \triangleq \mathrm{Cn}_{\mathcal{D}_{\veebar}} \subseteq C' \triangleq \mathrm{Cn}_{\mathcal{C}}$ is $\veebar$-disjunctive, and so, by Lemma 2.1, is δ-multiplicative. Then, the "if" part is by (2.3), (2.4) and (2.1) with $X = \varnothing$, $a = v$ and $Y = \{\phi/\psi\}$, for $C \subseteq C'$. Conversely, assume both $(\varphi \veebar v) \in C'(\phi \veebar v)$ and $(\varphi \veebar v) \in C'(\psi \veebar v)$, applying $[v/(\psi \veebar v)]$ and $[v/(v \veebar \varphi)]$, respectively, to which, by the structurality of C', we get both $(\varphi \veebar (\psi \veebar v)) \in C'(\phi \veebar (\psi \veebar v))$ and $(\varphi \veebar (v \veebar \varphi)) \in C'(\psi \veebar (v \veebar \varphi))$. In this way, as $C \subseteq C'$, by (2.1) with $X = \varnothing$, $a = v$ and $Y = \{\varphi \veebar \varphi\}$, (2.5), (2.6) and (2.7), we eventually get $(\varphi \veebar v) \in C'((\varphi \veebar \varphi) \veebar v) = C'(v \veebar (\varphi \veebar \varphi)) = C'((v \veebar \varphi) \veebar \varphi) = C'(\varphi \veebar (v \veebar \varphi)) \subseteq C'(\psi \veebar (v \veebar \varphi)) = C'((\psi \veebar v) \veebar \varphi) = C'(\varphi \veebar (\psi \veebar v)) \subseteq C'(\phi \veebar (\psi \veebar v)) = C'((\phi \veebar \psi) \veebar v)$, as required. □

4.2. Single- versus multi-conclusion sequent calculi

Let $\mathcal{G}^{\alpha}_{\veebar}$, where $\alpha \subseteq \omega$, be the α-conclusion sequent Σ-calculus constituted by structural α-conclusion sequent rules and the following α-conclusion sequent Σ-rules:

$$G_l \qquad\qquad G_r$$

$$\frac{\Gamma, x_0 \vdash \Delta \quad \Gamma, x_1 \vdash \Delta}{\Gamma, (x_0 \veebar x_1) \vdash \Delta} \qquad \frac{\Gamma \vdash \Omega, x_k}{\Gamma \vdash \Omega, (x_0 \veebar x_1)}$$

where $k \in 2$ and $\Gamma, \Delta, \Omega \in V^*_\omega$ such that $(\mathrm{dom}\,\Delta), ((\mathrm{dom}\,\Omega) + 1) \in \alpha$.

Lemma 4.6. *Let $\psi \in \mathrm{Fm}^{\omega}_{\veebar}$ and $v \in \mathrm{Var}(\psi)$. Suppose $1 \in \alpha$. Then, $v \vdash \psi$ is derivable in $\mathcal{G}^{\alpha}_{\veebar}$.*

Proof. By induction on construction of ψ. For consider the following complementary cases:

(1) $\psi \in V_\omega$.
Then, $\mathrm{Var}(\psi) = \{\psi\} \ni v$, in which case $\psi = v$, and so the Reflexivity axiom completes the argument.

(2) $\psi \notin V_\omega$.
Then, $\psi = (\varphi_0 \veebar \varphi_1)$, for some $\varphi_0, \varphi_1 \in \mathrm{Fm}^{\omega}_{\veebar}$, in which case $v \in \mathrm{Var}(\psi) = (\bigcup_{k \in 2} \mathrm{Var}(\varphi_k))$, and so $v \in \mathrm{Var}(\varphi_k)$, for some $k \in 2$. Hence, by induction hypothesis, $v \vdash \varphi_k$ is derivable in $\mathcal{G}^{\alpha}_{\veebar}$. In this way, G_r completes the argument. □

Corollary 4.7. *Let $\phi, \psi \in \mathrm{Fm}^{\omega}_{\veebar}$. Suppose $\mathrm{Var}(\phi) \subseteq \mathrm{Var}(\psi)$ and $1 \in \alpha$. Then, $\phi \vdash \psi$ is derivable in $\mathcal{G}^{\alpha}_{\veebar}$.*

Proof. By induction on construction of ϕ. For consider the following complementary cases:

(1) $\phi \in V_\omega$.
Then, $\mathrm{Var}(\psi) \supseteq \mathrm{Var}(\phi) = \{\phi\}$, in which case $\phi \in \mathrm{Var}(\psi)$, and so Lemma 4.6 completes the argument.

(2) $\phi \notin V_\omega$.
Then, $\phi = (\varphi_0 \veebar \varphi_1)$, for some $\varphi_0, \varphi_1 \in \mathrm{Fm}^\omega_\veebar$, in which case $\mathrm{Var}(\psi) \supseteq \mathrm{Var}(\phi) = (\bigcup_{k\in 2} \mathrm{Var}(\varphi_k))$, and so $\mathrm{Var}(\psi) \supseteq \mathrm{Var}(\varphi_k)$, for each $k \in 2$. Hence, by induction hypothesis, $\varphi_k \vdash \psi$ is derivable in $\mathcal{G}^\alpha_\veebar$, for every $k \in 2$. Thus, G_l completes the argument. □

Let $\tau_\veebar : \mathrm{Seq}^\omega_\Sigma \to \mathrm{Seq}^2_\Sigma$ be defined as follows:

$$\tau_\veebar(\Gamma \vdash \Delta) \triangleq \begin{cases} \Gamma \vdash \Delta & \text{if } \Delta = \varnothing, \\ \Gamma \vdash (\veebar\Delta) & \text{otherwise,} \end{cases}$$

for all $(\Gamma \vdash \Delta) \in \mathrm{Seq}^\omega_\Sigma$, in which case:

$$\sigma(\tau_\veebar(\Gamma \vdash \Delta)) = \tau_\veebar(\sigma(\Gamma \vdash \Delta)). \tag{4.6}$$

Theorem 4.8. *For every $\mathcal{R} \in \mathcal{G}^{\omega[\setminus 1]}_\veebar$, $\tau_\veebar(\mathcal{R})$ is derivable in $\mathcal{G}^{2[\setminus 1]}_\veebar$.*

Proof. Consider the following exhaustive cases:

(1) $\mathcal{R}$ is either G_l or the Reflexivity axiom or a left-side basic structural rule or a Cut with $\Delta = \varnothing$.
Then, $\tau_\veebar(\mathcal{R})$ is a substitutional Σ-instance of a rule in $\mathcal{G}^{2[\setminus 1]}_\veebar$, and so is derivable in it.

(2) $\mathcal{R}$ is either G_r or a right-side basic structural rule.
Then, $\tau_\veebar(\mathcal{R})$ is of the form

$$\frac{\Lambda \vdash \phi}{\Lambda \vdash \psi},$$

where $\Lambda \in V^*_\omega$ and $\phi, \psi \in \mathrm{Fm}^\omega_\veebar$, while $\mathrm{Var}(\phi) \subseteq \mathrm{Var}(\psi)$, in which case Corollary 4.7 and Cut complete the argument.

(3) $\mathcal{R}$ is a Cut with $\Delta \neq \varnothing$.
Then, $\tau_\veebar(\mathcal{R})$ is as follows:

$$\frac{\Lambda, \Gamma \vdash (\phi \veebar x_0) \quad \Gamma, x_0 \vdash \psi}{\Lambda, \Gamma \vdash \psi},$$

where $\phi \triangleq (\veebar\Delta) \in \mathrm{Fm}^\omega_\veebar$ and $\psi \triangleq (\veebar(\Delta, \Theta)) \in \mathrm{Fm}^\omega_\veebar$, in which case $\mathrm{Var}(\phi) \subseteq \mathrm{Var}(\psi)$, and so, by Corollary 4.7, $\phi \vdash \psi$ is derivable in $\mathcal{G}^{2[\setminus 1]}_\veebar$, and so is $\Gamma, \phi \vdash \psi$, by basic structural rules. Hence, by G_l, the rule $(\Gamma, x_0 \vdash \psi)|(\Gamma, (\phi \veebar x_0) \vdash \psi)$ is derivable in $\mathcal{G}^{2[\setminus 1]}_\veebar$. Thus, Cut completes the argument. □

Using induction on the length of $(\mathcal{G}^{\omega[\setminus 1]}_\veebar \cup \mathcal{A})$-derivations, by (4.6), Theorem 4.8 and the structurality of the consequence of any calculus, we immediately get:

Corollary 4.9. *Let* $(\mathcal{A} \cup \{\Phi\}) \subseteq \mathrm{Seq}_{\Sigma}^{\omega[\setminus 1]}$. *Suppose* Φ *is derivable in* $\mathcal{G}_{\veebar}^{\omega[\setminus 1]} \cup \mathcal{A}$. *Then,* $\tau_{\veebar}(\Phi)$ *is derivable in* $\mathcal{G}_{\veebar}^{2[\setminus 1]} \cup \tau_{\veebar}[\mathcal{A}]$.

5. Main results

Fix any finite $\veebar$-disjunctive Σ-matrix $\mathcal{A}$ with a finite equality determinant $\Upsilon \ni x_0$ to be supposed to be totally-ordered, x_0 being its greatest element. Given any $X \subseteq V_\omega$, put $\Upsilon[X] \triangleq \{\upsilon(x) \mid \upsilon \in \Upsilon, x \in X\}$.

To simplify further notations, we adopt the following "sign" one: given any $\Gamma \in (\mathrm{Fm}_{\Sigma}^{\omega})^*$ and any $\Bbbk \in 2$, put $(\Bbbk : \Gamma) \triangleq \{\langle \Bbbk, \Gamma\rangle, \langle 1 - \Bbbk, \varnothing\rangle\} \in \mathrm{Seq}_{\Sigma}^{\omega}$.

Following Pynko (2004), elements of $\Upsilon \times \Sigma$ are referred to as $\langle \Upsilon, \Sigma\rangle$-*types*, a $\langle \Upsilon, \Sigma\rangle$-type $\langle \upsilon, F\rangle$, where F is of arity $n \in \omega$, being said to be Υ-*complex*, whenever both $n \neq 0$ and $(n = 1) \Rightarrow (\upsilon(F(x_0)) \notin \Upsilon)$. Then, extending Pynko (2004), a Σ-*sequential* Υ-*table for* $\mathcal{A}$ is any couple $\mathcal{T}$ of functions with domain $\Upsilon \times \Sigma$, in which case we set $(\lambda/\rho)_{\mathcal{T}} \triangleq \pi_{0/1}(\mathcal{T})$ to adapt conventions adopted in Pynko (2004), such that, for all $\Bbbk \in 2$ and each $\langle \upsilon, F\rangle \in (\Upsilon \times \Sigma)$, where F is of arity $n \in \omega$, $\pi_{\Bbbk}(\mathcal{T})(\upsilon, F) \in \wp_\omega(\Upsilon[V_n]^*)^2$ has solely injective elements and is equivalent to $\Bbbk : \upsilon(F(x_i)_{i\in n})$ with respect to $\mathcal{A}$, that is, it holds that:

$$\mathcal{A} \models \langle \forall x_i\rangle_{i\in n}((\Bbbk : \upsilon(F(x_i)_{i\in n})) \leftrightarrow \pi_{\Bbbk}(\mathcal{T})(\upsilon, F)), \tag{5.1}$$

in which case every element of $(\boldsymbol{\lambda}/\boldsymbol{\rho})_{\mathcal{T}}(\upsilon, F) \triangleq ((\rho/\lambda)_{\mathcal{T}}(\upsilon, F) \uplus \{(0/1) : \upsilon(F(x_i)_{i\in n})\})$ is true in $\mathcal{A}$, that exists, by the constructive proof of Theorem 1 of Pynko (2004), though not being unique, generally speaking.

Example 5.1. When $\upsilon = x_0$ and $F = \veebar$, in which case $\veebar$ is a primary connective of Σ, one can always take $\lambda_{\mathcal{T}}(\upsilon, F) = \{x_0 \vdash; x_1 \vdash\}$ and $\rho_{\mathcal{T}}(\upsilon, F) = \{\vdash x_0, x_1\}$ to satisfy (5.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(\upsilon, F) = \{(x_0 \veebar x_1) \vdash x_0, x_1\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(\upsilon, F) = \{x_0 \vdash (x_0 \veebar x_1); x_1 \vdash (x_0 \veebar x_1)\}$, and so their elements are all derivable in $\mathcal{G}_{\veebar}^{\omega}$. □

Let A be a non-empty set of submatrices of $\mathcal{A}$, being uniquely determined by and so naturally identified with the carriers of their underlying algebras, in which case $m \triangleq |\mathsf{A}| \in (\omega \setminus 1)$, and $\overline{\mathcal{B}} : m \to \mathsf{A}$ any bijection.

Then, let $\mathcal{A}'$ be the set of all elements of $\boldsymbol{\lambda}_{\mathcal{T}}(\upsilon, F) \cup \boldsymbol{\rho}_{\mathcal{T}}(\upsilon, F)$, for all Υ-complex $\langle \Upsilon, \Sigma\rangle$-types $\langle \upsilon, F\rangle$ but $\langle x_0, \veebar\rangle$, in case $\veebar \in \Sigma$ is primary.

Next, let $\mathcal{A}''$ be the set containing, for each nullary $c \in \Sigma$ and every $\upsilon \in \Upsilon$, exactly that of the either axioms $(\upsilon(c) \vdash)/(\vdash \upsilon(c))$, which is true in $\mathcal{A}$.

Further, let $\mathcal{A}'''_j$, where $j \in m$, be the the set of all minimal (under $\sqsubseteq$) elements of the finite set of all those elements of $((\Upsilon)^*)^2$, which are both injective, disjoint and true in $\mathcal{B}_j$ as well as have monotonic sides, being clearly partially ordered by $\sqsubseteq$. Then, every element of $\mathcal{A}''' \triangleq \{\uplus\langle \Omega[x_0/x_j]\rangle_{j\in m} \mid \overline{\Omega} \in \prod_{j\in m} \mathcal{A}'''_j\}$ is true in A.

Finally, every element of $\mathcal{A} \triangleq (\mathcal{A}' \cup \mathcal{A}'' \cup \mathcal{A}''')$ is true in A. Moreover, $\mathcal{A}$ is finite, whenever Σ is so.

Lemma 5.2. *Any multi-conclusion* Σ*-sequent* Φ *is true in* A *iff it is derivable in* $\mathcal{G}_{\veebar}^{\omega} \cup \mathcal{A}$.

Proof. The "if" part is by the fact that every element of $\mathcal{A}$ is true in A, while any $\veebar$-disjunctive Σ-matrix (in particular, a submatrix of $\mathcal{A}$) is a model of $\mathcal{G}_{\veebar}^{\omega}$.

Conversely, assume Φ is true in A. Its derivability in $\mathcal{G}^{\omega}_{\underline{\vee}} \cup \mathcal{A}$ is proved by induction on $\partial(\Phi) \in \omega$, following the proof of Theorem 2 of Pynko (2004).

First, assume $\partial(\Phi) = 0$. The case, when Φ is not disjoint, is by Reflexivity and basic structural rules, Likewise, the case, when $\Psi \sqsubseteq \Phi$, for some $\Psi \in \mathcal{A}''$, is by basic structural rules. Otherwise, according to the item 4 of the proof of Theorem 2 of Pynko (2004), for each $j \in m$, as Φ is true in $\mathcal{B}_j$, there are some $v_j \in V_\omega$ and some $\Omega_j \in \mathcal{A}'''_j$ such that $(\Omega_j[x_0/v_j]) \sqsubseteq \Phi$, in which case $\Xi \triangleq (\uplus\langle\Omega[x_0/x_j]\rangle_{j\in m}) \in \mathcal{A}'''$ and $(\Xi[x_j/v_j]_{j\in m}) \sqsubseteq \Phi$, and so the derivability of Φ in $\mathcal{G}^{\omega}_{\underline{\vee}} \cup \mathcal{A}$ is by the structurality of the consequence of any calculus and basic structural rules.

Next, consider any complex $\langle \Upsilon, \Sigma\rangle$-type $\langle v, F\rangle$. We start from proving the fact that the rule:

$$\frac{\lambda_{\mathcal{T}}(v, F) \cup \rho_{\mathcal{T}}(v, F)}{\vdash} \tag{5.2}$$

is derivable in $\mathcal{G}^{\omega}_{\underline{\vee}} \cup \mathcal{A}$. Let $n \triangleq |\lambda_{\mathcal{T}}(v, F) \cup \rho_{\mathcal{T}}(v, F)| \in \omega$. Take any bijection $\overline{\Psi} : n \to (\lambda_{\mathcal{T}}(v, F) \cup \rho_{\mathcal{T}}(v, F))$. Then, by (5.1), the rule (5.2) is true in A, and so are all axioms in $(\overline{\Psi} \sqsupset \{\vdash\}) \subseteq (\Upsilon[V_\omega]^*)^2 \subseteq \partial^{-1}[\{0\}]$. Therefore, taking the above argumentation into account, all axioms in $(\overline{\Psi} \sqsupset \{\vdash\})$ are derivable in $\mathcal{G}^{\omega}_{\underline{\vee}} \cup \mathcal{A}$. Hence, applying n times Theorem 3.9, we conclude that the rule (5.2) is derivable in $\mathcal{G}^{\omega}_{\underline{\vee}} \cup \mathcal{A}$. Moreover, $\mathcal{G}^{\omega}_{\underline{\vee}} \cup \mathcal{A}$ is clearly multiplicative, and so deductively so. In this way, since every element of $\boldsymbol{(\lambda/\rho)}_{\mathcal{T}}(v, F)$, being in $\mathcal{A}$, unless $v = x_0$ and $F = \underline{\vee}$, is derivable in $\mathcal{G}^{\omega}_{\underline{\vee}} \cup \mathcal{A}$, in view of Example 5.1, taking basic structural rules into account, we see that the rule

$$\frac{(\lambda/\rho)_{\mathcal{T}}(v, F)}{(0/1) : v(F(x_i)_{i\in n})}$$

is derivable in $\mathcal{G}^{\omega}_{\underline{\vee}} \cup \mathcal{A}$. Thus, in view of the deductive multiplicativity of $\mathcal{G}^{\omega}_{\underline{\vee}} \cup \mathcal{A}$ as well as the structurality of the consequence of any calculus, taking basic structural rules into account, we see that all those rules, which belong to Definition 1(v) of Pynko (2004), are derivable in $\mathcal{G}^{\omega}_{\underline{\vee}} \cup \mathcal{A}$. In this way, the case, when $\partial(\Phi) \neq 0$, is due to the last paragraph of the proof of Theorem 2 of Pynko (2004), as required. □

Given any $\mathcal{B} \subseteq \mathrm{Seq}^{\omega}_{\Sigma}$, set $\mathcal{B}_{\backslash 1} \triangleq ((\mathcal{B} \cap \mathrm{Seq}^{\omega\backslash 1}_{\Sigma}) \cup \{(\sigma_{+1} \circ \Gamma) \vdash x_0 \mid \Gamma \in (\mathrm{Fm}^{\omega}_{\Sigma})^*, (\Gamma \vdash\ ) \in \mathcal{B}\}) \subseteq \mathrm{Seq}^{\omega\backslash 1}_{\Sigma}$. Clearly, elements of $\mathcal{A}_{\backslash 1}$ are true in A, for those of $\mathcal{A}$ are so.

Lemma 5.3. *Any purely multi-conclusion Σ-sequent is derivable in $\mathcal{G}^{\omega}_{\underline{\vee}} \cup \mathcal{A}$ iff it is derivable in $\mathcal{G}^{\omega\backslash 1}_{\underline{\vee}} \cup \mathcal{A}_{\backslash 1}$.*

Proof. The "if" part is by Lemma 5.2, for elements of $\mathcal{A}_{\backslash 1}$ are true in $\mathcal{A}$ being a model of $\mathcal{G}^{\omega\backslash 1}_{\underline{\vee}}$, for it is $\underline{\vee}$-disjunctive.

Conversely, consider any $\Phi = (\Gamma \vdash \Delta) \in \mathrm{Seq}^{\omega\backslash 1}_{\Sigma}$ and any $\mathcal{G}^{\omega}_{\underline{\vee}} \cup \mathcal{A}$-derivation $\overline{\Psi}$ of it of length $n \in \omega$. Take any $\varphi \in (\mathrm{img}\,\Delta) \neq \varnothing$. Then, in view of right-side basic structural rules, $\langle\langle \Psi_i \uplus (\vdash \varphi)\rangle_{i\in n}, \Phi\rangle$ is a $\mathrm{Cn}_{\mathcal{G}^{\omega\backslash 1}_{\underline{\vee}} \cup \mathcal{A}_{\backslash 1}}$-derivation of Φ, as required. □

Corollary 5.4. *Any [purely] single-conclusion Σ-sequent is true in A iff it is derivable in $\mathcal{G}^{2[\backslash 1]}_{\underline{\vee}} \cup \tau_{\underline{\vee}}[\mathcal{A}_{[\backslash 1]}]$.*

Proof. The "if" part is by the fact that every member of A, being a $\veebar$-disjunctive model of $\mathcal{A}_{[\setminus 1]}$, is then a model of $\mathcal{G}^{2[\setminus 1]}_{\veebar} \cup \tau_{\veebar}[\mathcal{A}_{[\setminus 1]}]$. The converse is by Corollary 4.9 and Lemma[s] 5.2 [and 5.3] as well as the diagonality of $\tau_{\veebar}{\restriction}\,\mathrm{Seq}^{2[\setminus 1]}_{\Sigma}$. $\square$

Given an axiomatic [finite] purely single-conclusion sequent Σ-calculus $\mathcal{G}$, we have the [finite] Hilbert-style Σ-calculus $(\mathcal{G}{\downarrow}) \triangleq \{(\mathrm{img}\,\Gamma)|\varphi \mid (\Gamma \vdash \varphi) \in \mathcal{G}\}$. Conversely, given a Hilbert-style Σ-calculus $\mathcal{C}$, we have the axiomatic purely single-conclusion sequent Σ-calculus $(\mathcal{C}{\uparrow}) \triangleq \{(\Gamma \vdash \varphi) \in \mathrm{Seq}^{2\setminus 1}_{\Sigma} \mid ((\mathrm{img}\,\Gamma)|\varphi) \in \mathcal{C}\}$, in which case $(\mathcal{C}{\uparrow}{\downarrow}) = \mathcal{C}$. Set $\mathcal{H} \triangleq ((\mathcal{D}_{\veebar} \cup (\tau_{\veebar}[\mathcal{A}] \cap \mathrm{Seq}^{0\vdash(2\setminus 1)}_{\Sigma}){\downarrow}) \cup (\sigma_{+1}[(\tau_{\veebar}[\mathcal{A}] \cap \mathrm{Seq}^{(\omega\setminus 1)\vdash(2\setminus 1)}_{\Sigma}){\downarrow}] \veebar x_0) \cup \{(\sigma_{+1}[\mathrm{img}\,\Gamma] \veebar x_0)|x_0 \mid \Gamma \in (\mathrm{Fm}^{\omega}_{\Sigma})^*, (\Gamma \vdash) \in \tau_{\veebar}[\mathcal{A}]\})$, being finite, whenever Σ is so, for $\mathcal{A}$ is then so.

Theorem 5.5. *The logic of* A *is axiomatized by* $\mathcal{H}$.

Proof. First of all, recall that $C \triangleq \mathrm{Cn}_{\mathcal{D}_{\veebar}}$ is $\veebar$-disjunctive (cf. Theorem 4.4), in which case, in particular, it satisfies (2.3), (2.5), (2.6) and (2.7), and so, for any $\Gamma \in \wp_{\omega}(\mathrm{Fm}^{\omega}_{\Sigma})$, any extension of C satisfies $(\sigma_{+1}[\Gamma] \veebar x_0)|x_0$ iff it satisfies $(\sigma_{+1}[\sigma_{+1}[\Gamma]] \veebar x_0)|(x_1 \veebar x_0)$. Therefore, $C' \triangleq \mathrm{Cn}_{\mathcal{H}}$ is equally axiomatized by $\mathcal{C}' \triangleq (\mathcal{D}_{\veebar} \cup (\mathcal{C} \cap \mathrm{Fm}^{\omega}_{\Sigma}) \cup (\sigma_{+1}[\mathcal{C} \setminus \mathrm{Fm}^{\omega}_{\Sigma}] \veebar x_0))$, where $\mathcal{C} \triangleq (\tau_{\veebar}[\mathcal{A}_{\setminus 1}]{\downarrow})$.

Next, every member of A, being a $\veebar$-disjunctive model of $\mathcal{A}_{\setminus 1}$, is so of $\tau_{\veebar}[\mathcal{A}_{\setminus 1}]$, and so of $\mathcal{C}$, and so of $\mathcal{C}'$, in view of Lemma 3.1.

Conversely, consider any Σ-rule $\mathcal{R} = (X|\varphi)$ true in A. Take any bijection $\Gamma : |X| \to X$. Then, the purely single-conclusion Σ-sequent $\Phi \triangleq (\Gamma \vdash \varphi)$ is true in A, and so is derivable in $\mathcal{G}^{2\setminus 1}_{\veebar} \cup \tau_{\veebar}[\mathcal{A}_{\setminus 1}]$, in view of Corollary 5.4. On the other hand, by Corollary 3.4, C' is $\veebar$-disjunctive. Let S be the set of all rules satisfied in C' (viz., derivable in $\mathcal{H}$, i.e., in $\mathcal{C}'$), in which case $\mathcal{C} \subseteq S$, by (2.3) and (2.5), and so $\tau_{\veebar}[\mathcal{A}_{\setminus 1}] \subseteq T \triangleq (S{\uparrow})$. Therefore, in view of the structurality and $\veebar$-disjunctivity of C', T is $(\mathcal{G}^{2\setminus 1}_{\veebar} \cup \tau_{\veebar}[\mathcal{A}_{\setminus 1}])$-closed. Hence, T contains all those purely single-conclusion Σ-sequents, which are derivable in $\mathcal{G}^{2\setminus 1}_{\veebar} \cup \tau_{\veebar}[\mathcal{A}_{\setminus 1}]$ (in particular, Φ). Thus, $\mathcal{R} \in (T{\downarrow}) = S$, as required. $\square$

5.1. Implicative case

Here, $\mathcal{A}$ is supposed to be a finite $\rhd$-implicative Σ-matrix with equality determinant $\Upsilon \ni x_0$, in which case it is $\veebar$-disjunctive, where $\veebar \triangleq \veebar_{\rhd}$ is *not* primary, and so is properly covered by the above discussion. Let $\tau_{\rhd} : \mathrm{Seq}^{2\setminus 1}_{\Sigma} \to \mathrm{Fm}^{\omega}_{\Sigma}, (\Gamma \vdash \phi) \mapsto (\Gamma \rhd \phi)$.

Example 5.6. When $\upsilon = x_0$ and $F = \rhd$, in which case $\rhd$ is a primary connective of Σ, one can always take $\lambda_{\mathcal{T}}(\upsilon, F) = \{\vdash x_0; x_1 \vdash\}$ and $\rho_{\mathcal{T}}(\upsilon, F) = \{x_0 \vdash x_1\}$ to satisfy (5.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(\upsilon, F) = \{x_0, (x_0 \rhd x_1) \vdash x_1\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(\upsilon, F) = \{\vdash x_0, (x_0 \rhd x_1); x_1 \vdash (x_0 \rhd x_1)\}$, and so elements of both $\tau_{\rhd}[\tau_{\veebar}[\boldsymbol{\lambda}_{\mathcal{T}}(\upsilon, F)]] = \{x_0 \rhd ((x_0 \rhd x_1) \rhd x_1)\}$ and $\tau_{\rhd}[\tau_{\veebar}[\boldsymbol{\rho}_{\mathcal{T}}(\upsilon, F)]] = \{(x_0 \rhd (x_0 \rhd x_1)) \rhd (x_0 \rhd x_1), (3.4)[x_0/x_1, x_1/x_0]\}$ are derivable in $\mathcal{I}_{\rhd}$, in view of Lemma 3.6, (3.1), (3.2) and (3.4). $\square$

In this way, let $\mathcal{A}'_{[\not\rhd]}$ be the set of all elements of $\boldsymbol{\lambda}_{\mathcal{T}}(\upsilon, F) \cup \boldsymbol{\rho}_{\mathcal{T}}(\upsilon, F)$, for all Υ-complex $\langle \Upsilon, \Sigma \rangle$-types $\langle \upsilon, F \rangle$ [but $\langle x_0, \rhd \rangle$, in case $\rhd \in \Sigma$ is primary]. Then, set $\mathcal{A}_{[\not\rhd]} \triangleq (\mathcal{A}'_{[\not\rhd]} \cup \mathcal{A}'' \cup \mathcal{A}''')$ and $\mathcal{I}_{[\not\rhd]} \triangleq (\mathcal{I}^{\mathrm{PL}}_{\rhd} \cup \tau_{\rhd}[\tau_{\veebar}[\mathcal{A}_{[\not\rhd]\setminus 1}]])$.

Theorem 5.7. *The logic of* A *is axiomatized by* $\mathcal{I}_{\not\rhd}$.

Proof. First of all, note that $C \triangleq \mathrm{Cn}_{\mathcal{I}_{\not\rhd}}$ is equally axiomatized by $\mathcal{I}$, in view of

Example 5.6, and is $\veebar$-disjunctive, by Theorem 3.7.

Next, every member of A, being an $\rhd$-implicative (in particular, $\veebar$-disjunctive) model of $\mathcal{A}_{\setminus 1}$, is so of $\tau_{\veebar}[\mathcal{A}_{\setminus 1}]$, and so of $\mathcal{I}$.

Conversely, consider any Σ-rule $\mathcal{R} = (X|\varphi)$ true in A. Take any bijection $\Gamma : |X| \to X$. Then, the purely single-conclusion Σ-sequent $\Phi \triangleq (\Gamma \vdash \varphi)$ is true in A, and so is derivable in $\mathcal{G}_{\veebar}^{2\setminus 1} \cup \tau_{\veebar}[\mathcal{A}_{\setminus 1}]$, in view of Corollary 5.4. Let S be the set of all rules satisfied in C (viz., derivable in $\mathcal{I}_{\not\rhd}$, i.e., in $\mathcal{I}$), in which case $\mathcal{I} \subseteq S$, and so, by (3.2), $\tau_{\veebar}[\mathcal{A}_{\setminus 1}] \subseteq T \triangleq (S\uparrow)$. Therefore, in view of the structurality and $\veebar$-disjunctivity of C, T is $(\mathcal{G}_{\veebar}^{2\setminus 1} \cup \tau_{\veebar}[\mathcal{A}_{\setminus 1}])$-closed. Hence, T contains all those purely single-conclusion Σ-sequents, which are derivable in $\mathcal{G}_{\veebar}^{2\setminus 1} \cup \tau_{\veebar}[\mathcal{A}_{\setminus 1}]$ (in particular, Φ), so $\mathcal{R} \in (T\downarrow) = S$. □

6. Applications and examples

Here, we use Theorems 5.5 and 5.7 tacitly, following notations adopted in the previous section and supposing that $\mathsf{A} = \{\mathcal{A}\}$, unless otherwise specified.

6.1. Disjunctive and implicative positive fragments of the classical logic

Here, we deal with the signature $\Sigma_{+[01]}^{(\supset)} \triangleq (\{\wedge, \vee\}[\cup\{\bot, \top\}](\cup\{\supset\}))$. By $\mathfrak{D}_{n[01]}^{(\supset)}$, where $n(= 2) \in (\omega \setminus 1)$, we denote the $\Sigma_{+[01]}^{(\supset)}$-algebra such that $\mathfrak{D}_{n[01]}^{(\supset)} \restriction \Sigma_{+[01]}$ is the [bounded] distributive lattice given by the chain n ordered by ordinal inclusion (and $(i \supset^{\mathfrak{D}_{2[01]}^{\supset}} j) \triangleq (\max(1 - i, j)$, for all $i, j \in 2$). Then, the logic of the $\vee$-disjunctive (and $\supset$-implicative) $\mathcal{D}_{2[01]}^{(\supset)} \triangleq \langle \mathfrak{D}_{2[01]}^{(\supset)}, \{1\}\rangle$ with equality determinant $\Upsilon = \{x_0\}$ (cf. Example 1 of Pynko (2004)) is the $\Sigma_{+[01]}^{(\supset)}$-fragment of the classical logic. Throughout the rest of this subsection, it is supposed that $\Sigma \subseteq \Sigma_{+,01}^{(\supset)}$ and $\mathcal{A} = (\mathcal{D}_{2,01}^{(\supset)} \restriction \Sigma)$, in which case $\mathcal{A}''' = \varnothing$.

First, in case $\Sigma = \{\supset\}$, both $\mathcal{A}'_{\not\supset}$ and $\mathcal{A}''$ are empty, and so is $\mathcal{A}_{\not\supset}$. In this way, we have the following well-known result:

Corollary 6.1. *The $\{\supset\}$-fragment of the classical logic is axiomatized by $\mathcal{I}_{\supset}^{\mathrm{PL}}$.*

Likewise, in case $\Sigma = \{\vee\}$, both $\mathcal{A}'$ and $\mathcal{A}''$ are empty, and so is $\mathcal{A}$. In this way, we get:

Corollary 6.2. *The $\{\vee\}$-fragment of the classical logic is axiomatized by $\mathcal{D}_{\vee}$.*

Next, let $\Sigma = \Sigma_+$. Then, $\mathcal{A}'' = \varnothing$, while one can take $\lambda_{\mathcal{T}}(x_0, \wedge) = \{x_0, x_1 \vdash\}$ and $\rho_{\mathcal{T}}(x_0, \wedge) = \{\vdash x_0; \vdash x_1\}$ to satisfy (5.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(x_0, \wedge) = \{(x_0 \wedge x_1) \vdash x_0; (x_0 \wedge x_1) \vdash x_1\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(x_0, \wedge) = \{x_0, x_1 \vdash (x_0 \wedge x_1)\}$, and so $\mathcal{A} = \mathcal{A}' = \{(x_0 \wedge x_1) \vdash x_0; (x_0 \wedge x_1) \vdash x_1; x_0, x_1 \vdash (x_0 \wedge x_1)\}$. Thus, we get:

Corollary 6.3. *The Σ_+-fragment of the classical logic is axiomatized by the calculus $\mathcal{PC}_+$ resulted from $\mathcal{D}_{\vee}$ by adding the following rules:*

$$C_1 \quad \frac{(x_1 \wedge x_2) \vee x_0}{x_1 \vee x_0} \qquad C_2 \quad \frac{(x_1 \wedge x_2) \vee x_0}{x_2 \vee x_0} \qquad C_3 \quad \frac{x_1 \vee x_0; x_2 \vee x_0}{(x_1 \wedge x_2) \vee x_0}$$

It is remarkable that the calculus $\mathcal{PC}_{+}$ consists of seven rules, while that which was found in Dyrda and Prucnal (1980) has nine rules. This demonstrates the practical applicability of our generic approach (more precisely, its factual ability to result in really "good" calculi to be enhanced a bit more by replacing appropriate pairs of rules/premises with single ones upon the basis of Corollary 4.5 and rules C_i, where $i \in (4 \setminus 1)$, whenever it is possible, to be done below tacitly — "on the fly").

Likewise, let $\Sigma = \Sigma_{+}^{\supset}$. Then, $\mathcal{A}'' = \varnothing$, and so, taking Corollary 3.8(ii) and Example 5.1 into account, we have the following well-known result:

Corollary 6.4. *The $\Sigma_{+}^{\supset}$-fragment of the classical logic is axiomatized by the calculus $\mathcal{PC}_{+}^{\supset}$ resulted from $\mathcal{I}_{\supset}^{\mathrm{PL}}$ by adding the following axioms:*

$$(x_0 \wedge x_1) \supset x_i \qquad\qquad x_0 \supset (x_1 \supset (x_0 \wedge x_1))$$

$$x_i \supset (x_0 \vee x_1) \qquad\qquad (x_0 \supset x_2) \supset ((x_1 \supset x_2) \supset ((x_0 \vee x_1) \supset x_2))$$

where $i \in 2$.

Finally, let $\Sigma = \Sigma_{+,01}^{[\supset]}$, in which case $\mathcal{A}'$ is as above, while $\mathcal{A}'' = \{\vdash \top; \bot \vdash\}$, and so [taking Corollary 3.8(ii) into account] we get:

Corollary 6.5. *The $\Sigma_{+,01}^{[\supset]}$-fragment of the classical logic is axiomatized by the calculus $\mathcal{PC}_{+,01}^{[\supset]}$ resulted from $\mathcal{PC}_{+}^{[\supset]}$ by adding the following rules:*

$$\top \qquad\qquad \frac{\bot \vee x_0}{x_0}[\bot \supset x_0]$$

6.2. Miscellaneous four-valued expansions of Belnap's four-valued logic

Let $\Sigma_{\sim,+[01]}^{(\supset)} \triangleq (\Sigma_{+[01]}^{(\supset)} \cup \{\sim\})$, where $\sim$ (weak negation) is unary.

Here, it is supposed that $\Sigma \supseteq \Sigma_{\sim,+[01]}$, $(\mathfrak{A}{\restriction}\Sigma_{\sim,+[01]}) = \mathfrak{DM}_{4[01]}$, where $(\mathfrak{DM}_{4[01]}{\restriction}\Sigma_{+[01]}) \triangleq \mathfrak{D}_{2[01]}^{2}$, while $\sim^{\mathfrak{DM}_{4[01]}}\langle i, j\rangle \triangleq \langle 1-j, 1-i\rangle$, for all $i, j \in 2$, in which case we use the following standard notations going back to Belnap (1977):

$$\mathsf{t} \triangleq \langle 1,1\rangle, \qquad \mathsf{f} \triangleq \langle 0,0\rangle, \qquad \mathsf{b} \triangleq \langle 1,0\rangle, \qquad \mathsf{n} \triangleq \langle 0,1\rangle,$$

and $\mathcal{A} \triangleq \langle \mathfrak{A}, \{\mathsf{b}, \mathsf{t}\}\rangle$, in which case it is $\vee$-disjunctive, while $\Upsilon = \{x_0, \sim x_0\}$ is an equality determinant for it (cf. Example 2 of Pynko (2004)), whereas $\mathcal{A}''' = \varnothing$. (Since the logic $B_{4[01]}$ of $\mathcal{A}{\restriction}\Sigma_{\sim,+[01]}$ is the [bounded version of] Belnap's logic, the logic of $\mathcal{A}$ is a four-valued expansion of $B_{4[01]}$.)

First, let $\Sigma = \Sigma_{\sim,+}$, in which case $\mathcal{A}'' = \varnothing$, while the case of the Υ-complex $\langle \Upsilon, \Sigma\rangle$-type $\langle x_0, \wedge\rangle$ is as in the previous subsection, whereas others but $\langle x_0, \vee\rangle$ are as follows. First of all, one can take $\lambda_{\mathcal{T}}(\sim x_0, \vee) = \{\sim x_0, \sim x_1 \vdash\}$ and $\rho_{\mathcal{T}}(\sim x_0, \vee) = \{\vdash \sim x_0; \vdash \sim x_1\}$ to satisfy (5.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(\sim x_0, \vee) = \{\sim(x_0 \vee x_1) \vdash \sim x_0; \sim(x_0 \vee x_1) \vdash \sim x_1\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(\sim x_0, \vee) = \{\sim x_0, \sim x_1 \vdash \sim(x_0 \vee x_1)\}$. Likewise, one can take $\lambda_{\mathcal{T}}(\sim x_0, \wedge) = \{\sim x_0 \vdash; \sim x_1 \vdash\}$ and $\rho_{\mathcal{T}}(\sim x_0, \wedge) = \{\vdash \sim x_0, \sim x_1\}$ to satisfy (5.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(\sim x_0, \wedge) = \{\sim(x_0 \wedge x_1) \vdash \sim x_0, \sim x_1\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(\sim x_0, \wedge) = \{\sim x_0 \vdash \sim(x_0 \wedge x_1); \sim x_1 \vdash \sim(x_0 \wedge x_1)\}$. Finally, one can take $\lambda_{\mathcal{T}}(\sim x_0, \sim) = \{x_0 \vdash\}$ and $\rho_{\mathcal{T}}(\sim x_0, \sim) = \{\vdash x_0\}$ to satisfy (5.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(\sim x_0, \sim) = \{\sim\sim x_0 \vdash x_0\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(\sim x_0, \sim) = \{x_0 \vdash \sim\sim x_0\}$. In this way, we get:

Corollary 6.6. *B_4 is axiomatized by the calculus $\mathcal{B}$ resulted from $\mathcal{PC}_+$ by adding the following rules as well as the inverse to these:*

$$\begin{array}{ccc} NN & ND & NC \\[4pt] \dfrac{x_1 \vee x_0}{\sim\sim x_1 \vee x_0} & \dfrac{(\sim x_1 \wedge \sim x_2) \vee x_0}{\sim(x_1 \vee x_2) \vee x_0} & \dfrac{(\sim x_1 \vee \sim x_2) \vee x_0}{\sim(x_1 \wedge x_2) \vee x_0} \end{array}$$

The calculus $\mathcal{B}$ has 13 rules, while the very first axiomatization of B_4 discovered in Pynko (1995) (cf. Definition 5.1 and Theorem 5.2 therein)[2] has 15 rules, "two rules win" being just due to the advance of the present study with regard to Dyrda and Prucnal (1980) (cf. the previous subsection).

Now, let $\Sigma = \Sigma_{\sim,+,01}$, in which case both $\mathcal{A}'$ and $\mathcal{A}'''$ are as above, while $\mathcal{A}'' = \{\top; \sim\bot; \bot\vdash; \sim\top\vdash\}$, and so we get:

Corollary 6.7. *$B_{4,01}$ is axiomatized by the calculus $\mathcal{B}_{01}$ resulted from $\mathcal{B} \cup \mathcal{PC}_{+,01}$ by adding the following axiom and rule:*

$$\sim\bot \qquad\qquad \frac{\sim\top \vee x_0}{x_0}$$

6.2.1. The classical expansion

Let $\Sigma^{(\supset)}_{\simeq,+[01]} \triangleq (\Sigma^{(\supset)}_{\sim,+[01]} \cup \{\neg\})$, where $\neg$ (classical negation) is unary.

Here, it is supposed that $\Sigma = \Sigma_{\simeq,+[01]}$, while $\neg^{\mathfrak{A}}\langle i,j\rangle \triangleq \langle 1-i, 1-j\rangle$, for all $i,j \in 2$. Then, one can take $\lambda_{\mathcal{T}}(x_0,\neg) = \{\vdash x_0\}$ and $\rho_{\mathcal{T}}(x_0,\neg) = \{x_0 \vdash\}$ to satisfy (5.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(x_0,\neg) = \{x_0, \neg x_0 \vdash\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(x_0,\neg) = \{\vdash x_0, \neg x_0\}$. Likewise, one can take $\lambda_{\mathcal{T}}(\sim x_0,\neg) = \{\vdash \sim x_0\}$ and $\rho_{\mathcal{T}}(\sim x_0,\neg) = \{\sim x_0 \vdash\}$ to satisfy (5.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(\sim x_0,\neg) = \{\sim x_0, \sim\neg x_0 \vdash\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(\sim x_0,\neg) = \{\vdash \sim x_0, \sim\neg x_0\}$. Thus, we get:

Corollary 6.8. *The logic of $\mathcal{A}$ is axiomatized by the calculus $\mathcal{CB}_{[01]}$ resulted from $\mathcal{B}_{[01]}$ by adding the following rules:*

$$\begin{array}{cccc} N_1 & N_2 & N_3 & N_4 \\[4pt] \dfrac{(x_1 \wedge \neg x_1) \vee x_0}{x_0} & x_0 \vee \neg x_0 & \dfrac{(\sim x_1 \wedge \sim\neg x_1) \vee x_0}{x_0} & \sim x_0 \vee \sim\neg x_0 \end{array}$$

6.2.2. The bilattice expansions

Let $\Sigma^{(\supset)}_{\sim/\simeq,2:+[01]} \triangleq (\Sigma^{(\supset)}_{\sim/\simeq,+[01]} \cup \{\sqcap, \sqcup\}[\cup\{\mathbf{0},\mathbf{1}\}])$, where $\sqcap$ and $\sqcup$ (*knowledge* conjunction and disjunction, respectively) are binary [while $\mathbf{0}$ and $\mathbf{1}$ are nullary].

Here, it is supposed that $\Sigma = \Sigma_{\sim/\simeq,2:+[01]}$, while

$$(\langle i,j\rangle(\sqcap/\sqcup)^{\mathfrak{A}}\langle k,l\rangle) \triangleq \langle(\min/\max)(i,k), (\max/\min)(j,l)\rangle,$$

for all $i,j,k,l \in 2$ [whereas $\mathbf{0}^{\mathfrak{A}} \triangleq \mathsf{n}$ and $\mathbf{1}^{\mathfrak{A}} \triangleq \mathsf{b}$].

First, let $\Sigma = \Sigma_{\sim,2:+}$, in which case $\mathcal{A}'' = \varnothing$. Then, one can take $\lambda_{\mathcal{T}}(x_0,\sqcap) = \{x_0, x_1 \vdash\}$ and $\rho_{\mathcal{T}}(x_0,\sqcap) = \{\vdash x_0; \vdash x_1\}$ to satisfy (5.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(x_0,\sqcap) =$

[2] In this connection, we should like to take the opportunity to specify the ambiguous footnote 3 on p. 443 therein. The problem has been that, as we have noticed, because of missing a reservation like "in reply to our first informing him about this result two weeks before" just after "1994", the mentioned footnote has been misleading readers leaving them with wrong impression about the genuine priority/authorship as to this result.

$\{(x_0 \sqcap x_1) \vdash x_0; (x_0 \sqcap x_1) \vdash x_1\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(x_0, \sqcap) = \{x_0, x_1 \vdash (x_0 \sqcap x_1)\}$. Likewise, one can take $\lambda_{\mathcal{T}}(x_0, \sqcup) = \{x_0 \vdash; x_1 \vdash\}$ and $\rho_{\mathcal{T}}(x_0, \sqcup) = \{\vdash x_0, x_1\}$ to satisfy (5.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(x_0, \sqcup) = \{(x_0 \sqcup x_1) \vdash x_0, x_1\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(x_0, \sqcup) = \{x_0 \vdash (x_0 \sqcup x_1); x_1 \vdash (x_0 \sqcup x_1)\}$. Next, one can take $\lambda_{\mathcal{T}}(\sim x_0, \sqcap) = \{\sim x_0, \sim x_1 \vdash\}$ and $\rho_{\mathcal{T}}(\sim x_0, \sqcap) = \{\vdash \sim x_0; \vdash \sim x_1\}$ to satisfy (5.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(\sim x_0, \sqcap) = \{\sim(x_0 \sqcap x_1) \vdash \sim x_0; \sim(x_0 \sqcap x_1) \vdash \sim x_1\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(\sim x_0, \sqcap) = \{\sim x_0, \sim x_1 \vdash \sim(x_0 \sqcap x_1)\}$. Finally, one can take $\lambda_{\mathcal{T}}(\sim x_0, \sqcup) = \{\sim x_0 \vdash; \sim x_1 \vdash\}$ and $\rho_{\mathcal{T}}(\sim x_0, \sqcup) = \{\vdash \sim x_0, \sim x_1\}$ to satisfy (5.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(\sim x_0, \sqcup) = \{\sim(x_0 \sqcup x_1) \vdash \sim x_0, \sim x_1\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(\sim x_0, \sqcup) = \{\sim x_0 \vdash \sim(x_0 \sqcup \sim x_1); \sim x_1 \vdash \sim(x_0 \sqcup x_1)\}$. Thus, we get:

Corollary 6.9. *The logic of $\mathcal{A}$ is axiomatized by the calculus $\mathcal{BL}$ resulted from adding to $\mathcal{B}$ the following rules as well as the inverse to these:*

$$\begin{array}{cccc} KC & KD & NKC & NKD \\ \dfrac{(x_1 \wedge x_2) \vee x_0}{(x_1 \sqcap x_2) \vee x_0} & \dfrac{(x_1 \vee x_2) \vee x_0}{(x_1 \sqcup x_2) \vee x_0} & \dfrac{(\sim x_1 \wedge \sim x_2) \vee x_0}{\sim(x_1 \sqcap x_2) \vee x_0} & \dfrac{(\sim x_1 \vee \sim x_2) \vee x_0}{\sim(x_1 \sqcup x_2) \vee x_0} \end{array}$$

Likewise, let $\Sigma = \Sigma_{\sim,2+,01}$, in which case both $\mathcal{A}'$ and $\mathcal{A}'''$ are as above, while $\mathcal{A}'' = (\{\bot \vdash; \top\} \cup \{\sim^i \mathbf{0} \vdash; \sim^i \mathbf{1} \mid i \in 2\})$, and so we have:

Corollary 6.10. *The logic of $\mathcal{A}$ is axiomatized by the calculus $\mathcal{BL}_{01}$ resulted from adding to $\mathcal{BL} \cup \mathcal{B}_{01}$ the following axioms and rules:*

$$\sim^i \mathbf{1} \qquad\qquad \frac{\sim^i \mathbf{0} \vee x_0}{x_0}$$

where $i \in 2$.

Finally, when $\Sigma = \Sigma_{\simeq,2:+[01]}$, we have:

Corollary 6.11. *The logic of $\mathcal{A}$ is axiomatized by the calculus $\mathcal{CB} \cup \mathcal{BL}_{[01]}$.*

6.2.3. Implicative expansions

Here, it is supposed that $\supset \in \Sigma$, while $(\langle i, j\rangle \supset^{\mathfrak{A}} \langle k, l\rangle) \triangleq \langle \max(1-i, k), \max(1-i, l)\rangle$, for all $i, j, k, l \in 2$, in which case $\mathcal{A}$ is $\supset$-implicative.

First, let $\Sigma = \Sigma^{\supset}_{\sim,+}$. Clearly, one can take $\lambda_{\mathcal{T}}(\sim x_0, \supset) = \{x_0, \sim x_1 \vdash\}$ and $\rho_{\mathcal{T}}(\sim x_0, \supset) = \{\vdash x_0; \vdash \sim x_1\}$ to satisfy (5.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(\sim x_0, \supset) = \{\sim(x_0 \supset x_1) \vdash x_0; \sim(x_0 \supset x_1) \vdash \sim x_1\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(\sim x_0, \supset) = \{x_0, \sim x_1 \vdash \sim(x_0 \supset x_1)\}$. Therefore, taking Corollary 3.8(ii) and Example 5.1 into account, we get:

Corollary 6.12. *The logic of $\mathcal{A}$ is axiomatized by the calculus $\mathcal{B}^{\supset}$ resulted from $\mathcal{PC}^{\supset}_{+}$ by adding the following axioms:*

$$\begin{array}{llr} \sim\sim x_0 \supset x_0 & x_0 \supset \sim\sim x_0 & (6.1) \\ \sim(x_0 \vee x_1) \supset \sim x_i & \sim x_0 \supset (\sim x_1 \supset \sim(x_0 \vee x_1)) & (6.2) \\ \sim x_i \supset \sim(x_0 \wedge x_1) & (\sim x_0 \supset x_2) \supset ((\sim x_1 \supset x_2) \supset (\sim(x_0 \wedge x_1) \supset x_2)) & (6.3) \\ \sim(x_0 \supset x_1) \supset \sim^i x_i & x_0 \supset (\sim x_1 \supset \sim(x_0 \supset x_1)) & \end{array}$$

where $i \in 2$.

It is remarkable that $\mathcal{B}^{\supset}$ is actually the calculus *Par* introduced in Popov (1989) but regardless to any semantics. In this way, the present study provides a new (and quite immediate) insight into the issue of semantics of *Par* first being due to Pynko (1999) but with using the intermediate purely multi-conclusion sequent calculus *GPar* actually introduced in Popov (1989) regardless to any semantics too and then studied semantically in Pynko (1999).

Likewise, in case $\Sigma = \Sigma^{\supset}_{\sim,+,01}$, we have:

Corollary 6.13. *The logic of $\mathcal{A}$ is axiomatized by the calculus $\mathcal{B}^{\supset}_{01}$ resulted from $\mathcal{B}^{\supset} \cup \mathcal{PC}^{\supset}_{+,01}$ by adding the following axioms:*

$$\sim\bot \qquad\qquad \sim\top \supset x_0$$

Now, let $\Sigma = \Sigma^{\supset}_{\sim,2:+}$. Then, we have:

Corollary 6.14. *The logic of $\mathcal{A}$ is axiomatized by the calculus $\mathcal{BL}^{\supset}$ resulted from $\mathcal{B}^{\supset}$ by adding the following axioms:*

$$(x_0 \sqcap x_1) \supset x_i \qquad x_0 \supset (x_1 \supset (x_0 \sqcap x_1))$$
$$x_i \supset (x_0 \sqcup x_1) \qquad (x_0 \supset x_2) \supset ((x_1 \supset x_2) \supset ((x_0 \sqcup x_1) \supset x_2))$$
$$\sim(x_0 \sqcap x_1) \supset \sim x_i \qquad \sim x_0 \supset (\sim x_1 \supset \sim(x_0 \sqcap x_1))$$
$$\sim x_i \supset \sim(x_0 \sqcup x_1) \qquad (\sim x_0 \supset x_2) \supset ((\sim x_1 \supset x_2) \supset (\sim(x_0 \sqcup x_1) \supset x_2))$$

where $i \in 2$.

Likewise, when $\Sigma = \Sigma^{\supset}_{\sim,2:+,01}$, we have:

Corollary 6.15. *The logic of $\mathcal{A}$ is axiomatized by the calculus $\mathcal{BL}^{\supset}_{01}$ resulted from $\mathcal{BL}^{\supset} \cup \mathcal{B}^{\supset}_{01}$ by adding the following axioms:*

$$\sim^i \mathbf{1} \qquad\qquad \sim^i \mathbf{0} \supset x_0$$

where $i \in 2$.

Further, let $\Sigma = \Sigma^{\supset}_{\simeq,+[01]}$. Then, taking (3.2) and Corollary (3.8)(i) into account, we have:

Corollary 6.16. *The logic of $\mathcal{A}$ is axiomatized by the calculus $\mathcal{CB}^{\supset}_{[01]}$ resulted from $\mathcal{B}^{\supset}_{[01]}$ by adding the axioms N_2, N_4 and the following ones:*

$$\sim^i x_1 \supset (\sim^i \neg x_i \supset x_0),$$

where $i \in 2$.

Finally, when $\Sigma = \Sigma^{\supset}_{\simeq,2:+[01]}$, we have:

Corollary 6.17. *The logic of $\mathcal{A}$ is axiomatized by the calculus $\mathcal{CB}^{\supset} \cup \mathcal{BL}^{\supset}_{[01]}$.*

6.2.4. Three-valued extensions

In case (both of) $A_{\not{\mathsf{n}}/\not{\mathsf{b}}} \triangleq (A \setminus \{\mathsf{n}/\mathsf{b}\})$ forms a subalgebra of $\mathfrak{A}$, $\mathsf{A} \triangleq \{\mathcal{A} \restriction A_{\not{\mathsf{n}}/\not{\mathsf{b}}}\}$ $(\{\mathcal{A} \restriction A_{\not{\mathsf{n}}}, \mathcal{A} \restriction A_{\not{\mathsf{b}}}\})$ inherits both $\mathcal{A}'_{[\supset]}$ and $\mathcal{A}''$, but $\mathcal{A}''' = \{(1/0) : (\sim x_0, x_0)\}(\{\sim x_0, x_0 \vdash \sim x_1, x_1\})$, and so we get the following universal conclusion:

Corollary 6.18. *Suppose $\Sigma \supseteq \Sigma^{[\supset]}_{\sim,+}$ and (both of) $A_{\not{\mathsf{n}}/\not{\mathsf{b}}}$ forms a subalgebra of $\mathfrak{A}$. Then, the logic of A is axiomatized by the Σ-calculus resulted from any Σ-calculus axiomatizing the logic of $\mathcal{A}$ by adding the* Excluded Middle Law *axiom/the* Resolution *rule [resp., the* Ex Contradictione Quodlibet *axiom] $(\sim x_0 \vee x_0)/(((x_0 \vee x_1) \wedge (\sim x_0 \vee x_1))|x_1)[\sim x_1 \supset (x_1 \supset x_0)]$ (resp., the rule [without premises] $((x_1 \vee x_0) \wedge (\sim x_1 \vee x_0))|((\sim x_2 \vee x_2) \vee x_0)[\sim x_1 \supset (x_1 \supset (\sim x_0 \vee x_0))]$.*

This covers arbitrary three-valued expansions of the *logic of paradox LP* Priest (1979) {including *LP* itself, when $\Sigma = \Sigma_{\sim,+}$, and so subsuming Corollary 5.3 of Pynko (1995), its bounded expansion, when $\Sigma = \Sigma_{\sim,+,01}$, the *logic of antinomies LA* Asenjo and Tamburino (1975), when $\Sigma = \Sigma^{\supset}_{\sim,+}$, and *J*3 D'Ottaviano and Epstein (1988), when $\Sigma = \Sigma^{\supset}_{\sim,+,01}$ — up to term-wise definitional equivalence}/*Kleene's three-valued logic* K_3 Kleene (1952) (resp., $LP \cap K_3$). In particular, it appears that the $\Sigma^{\supset}_{\sim,+}$-calculus *Pcont* Popov (1989), resulted from *Par* by adducing the Excluded Middle Law axiom and involved therein regardless to any semantics as well, axiomatizes *LA*.

6.3. Łukasiewicz finitely-valued logics

Let $\Sigma \triangleq \{\supset, \neg\}$, $n \in (\omega \setminus 2)$ and $\mathcal{L}_n$ the Σ-matrix with $L_n \triangleq n$, $D^{\mathcal{L}_n} \triangleq \{n-1\}$, $\neg^{\mathfrak{L}_n} i \triangleq (n-1-i)$ and $(i \supset^{\mathfrak{L}_n} j) \triangleq \min(n-1, n-1-i+j)$, for all $i, j \in n$. The logic $Ł_n$ of $\mathcal{L}_n$ is known as *Łukasiewicz n-valued logic* (cf. Łukasiewicz (1920) for the three-valued case alone though). By induction on any $m \in (\omega \setminus 1)$, define the secondary unary connective $m\otimes$ of Σ as follows:

$$(m \otimes x_0) \triangleq \begin{cases} x_0 & \text{if } m = 1, \\ \neg x_0 \supset ((m-1) \otimes x_0) & \text{otherwise,} \end{cases}$$

in which case $(m \otimes^{\mathfrak{L}_n} i) = \min(n-1, m \cdot i)$, for all $i \in n$, and so, in particular, $(m\otimes)^{\mathfrak{L}_n}$ is monotonic. Then, set $(\Box x_0) \triangleq (\neg^{\min(1,n-2)}(n-1) \otimes \neg^{\min(1,n-2)} x_0)$ and $(x_0 \rhd x_1) \triangleq (\Box x_0 \supset \Box x_1)$, being secondary, unless $n = 2$, when $(\Box x_0) = x_0$, and so $\rhd = \supset$ is primary. In that case, $\Box^{\mathfrak{L}_n} = (((n-1) \times \{0\}) \cup \{\langle n-1, n-1 \rangle\})$, and so $\mathcal{L}_n$ is $\rhd$-implicative, for $\mathcal{L}_n \restriction \{0, n-1\}$ is $\supset$-implicative.

And what is more, according to the constructive proof of Proposition 6.10 of Pynko (2009), for each $i \in ((n-1) \setminus 2)$, there is some $\upsilon_i \in \mathrm{Fm}^1_{\{\neg, 2\otimes\}}$ such that $(\upsilon_i^{\mathfrak{L}_n}(i) = (n-1)) \Leftrightarrow (\upsilon_i^{\mathfrak{L}_n}(i-1) \neq (n-1))$. In addition, put $\upsilon_{n-1} \triangleq x_0 \in \mathrm{Fm}^1_{\{\neg, 2\otimes\}}$ and, in case $n \neq 2$, $\upsilon_1 \triangleq \neg x_0 \in \mathrm{Fm}^1_{\{\neg, 2\otimes\}}$. In this way, for each $i \in (n \setminus 1)$, it holds that $(\upsilon_i^{\mathfrak{L}_n}(i) = (n-1)) \Leftrightarrow (\upsilon_i^{\mathfrak{L}_n}(i-1) \neq (n-1))$. On the other hand, for every $\upsilon \in \mathrm{Fm}^1_{\{\neg, 2\otimes\}}$, $\upsilon^{\mathfrak{L}_n}$ is either monotonic or anti-monotonic, for both $x_0^{\mathfrak{L}_n} = \Delta_n$ and $(2\otimes)^{\mathfrak{L}_n}$ are monotonic, while $\neg^{\mathfrak{L}_n}$ is anti-monotonic. Therefore, for each $i \in N_{0/1} \triangleq \{j \in (n \setminus 1) \mid \upsilon_j^{\mathfrak{L}_n}(j) = / \neq (n-1)\}$, $\upsilon_i^{\mathfrak{L}_n}$ is monotonic/anti-monotonic, in which case $(\upsilon_j^{\mathfrak{L}_n})^{-1}[\{n-1\}] = ((n \setminus i)/i)$, and so $\Upsilon \triangleq \{\upsilon_i \mid i \in (n \setminus 1)\} \supseteq (\{x_0\} \cup \{\neg x_0 \mid n \neq 2\})$ is a finite equality determinant for $\mathcal{L}_n$, $\bar{\upsilon} : (n \setminus 1) \to \Upsilon$ being a bijection supposed to induce a total ordering on Υ, in which case $\langle x_0, \neg\rangle = \langle \nu_{n-1}, \neg\rangle$ is not Υ-complex, unless $n = 2$, when all $\langle\Upsilon, \Sigma\rangle$-types are Υ-complex, for, in that case, $\Upsilon = \{x_0\}$. And what is more, as it follows from the constructive proof of Proposition 6.10 of Pynko (2009), non-Υ-complex $\langle\Upsilon, \Sigma\rangle$-types other than $\langle x_0, \neg\rangle$ are exactly those of the form $\langle \upsilon_i, \neg\rangle$, where $\frac{n-1}{2} \geqslant i \in (n \setminus 2)$, and so a $\langle\Upsilon, \Sigma\rangle$-type of the form $\langle \upsilon_i, \neg\rangle$, where $i \in (n \setminus 1)$, is Υ-complex iff $i \in N \triangleq \{j \in ((n - \min(1, n-2)) \setminus 1) \mid (j \neq 1) \Rightarrow ((n-1) \in (2 \cdot j))\}$. In particular, in case $n \in (5 \setminus 3)$, $\langle x_0, \neg\rangle$ is the only non-Υ-complex $\langle\Upsilon, \Sigma\rangle$-type. As $(N_0 \cap N_1) = \varnothing$ and $(N_0 \cup N_1) = (n \setminus 1)$, we have the mapping $\mu \triangleq \{\langle i, k\rangle \in ((n \setminus 1) \times 2) \mid i \in N_k\} : (n \setminus 1) \to 2$.

Let $\mathcal{A} \triangleq \mathcal{L}_n$. Then, $\mathcal{A}'' = \varnothing$. Moreover, under the conventions adopted in both Pynko (2014) and Pynko (2015), we see that both

$$\begin{aligned} \{I_{i-1} : \varphi\} &\leftrightarrow (\mu(i) : \upsilon_i(\varphi)), \\ \{F_i : \varphi\} &\leftrightarrow ((1 - \mu(i)) : \upsilon_i(\varphi)), \end{aligned}$$

where $i \in (n \setminus 1)$ and $\varphi \in \mathrm{Fm}^{\omega}_{\Sigma}$, are true in $\mathcal{A}$. Hence, in view of Corollary 2.4 of Pynko (2014), $\mathcal{A}''' = \{((1 - \mu(i)) : \upsilon_i) \uplus (\mu(j) : \upsilon_j) \mid i, j \in (n \setminus 1), i \in j\}$. And what is more, taking Pynko (2014) into account, in view of Lemma 2.1 of Pynko (2015), we have a Σ-sequential Υ-table $\mathcal{T}$ for $\mathcal{A}$ given as follows. First, for all $i \in (n \setminus 1)$ and all $m \in 2$, let $\pi_m(\mathcal{T})(\upsilon_i, \neg) \triangleq \{(1-)^{\mu(i)}(1-)^m(1 - \mu(n-i)) : \upsilon_{n-i}\}$. Next, for all $i \in (n \setminus 1)$, let $\pi_{1-\mu(i)}(\mathcal{T})(\upsilon_i, \supset) \triangleq \{(\mu(n-1-k) : \nu_{n-1-k}) \uplus ((1 - \mu(i-k)) : \nu_{i-k}(x_1)) \mid k \in i\}$ and $\pi_{\mu(i)}(\mathcal{T})(\upsilon_i, \supset) \triangleq (\{((1 - \mu(n-k)) : \upsilon_{n-k}) \uplus (\mu(i-k) : \upsilon_{i-k}(x_1)) \mid k \in (i \setminus 1)\} \cup \{(1 - \mu(n-i)) : \upsilon_{n-i}; \mu(i) : \upsilon_i(x_1)\})$. In this way, taking Corollary 3.8(ii) into account, we eventually get:

Corollary 6.19. *$Ł_n$ is axiomatized by the finite calculus $\mathcal{L}_n$ resulted from $\mathfrak{I}^{\mathrm{PL}}_{\rhd}$ by adding the following axioms:*

$\upsilon_i \rhd \upsilon_j$	$(\langle i, j\rangle \in ((\ker \mu) \cap (\in \cap\, n^2)^{(2 \cdot \mu(i)) - 1})$
$\upsilon_i \veebar_{\rhd} \upsilon_j$	$(\langle i, j\rangle \in (\mu^{-1}[\in \cap\, 2^2] \cap (\in \cap\, n^2))$
$\upsilon_i \rhd (\upsilon_j \rhd x_1)$	$(\langle i, j\rangle \in (\mu^{-1}[\ni \cap\, 2^2] \cap (\in \cap\, n^2))$
$\upsilon_{n-i} \veebar_{\rhd} \upsilon_i(\neg x_0)$	$(i \in N, \mu(i) = \mu(n-i))$
$\upsilon_{n-i} \rhd (\upsilon_i(\neg x_0) \rhd x_1)$	$(i \in N, \mu(i) = \mu(n-i))$
$\upsilon_{n-i} \rhd \upsilon_i(\neg x_0)$	$(i \in N, \mu(i) \neq \mu(n-i))$
$\upsilon_i(\neg x_0) \rhd \upsilon_{n-i}$	$(i \in N, \mu(i) \neq \mu(n-i))$
$\upsilon_{n-1-k} \rhd (\upsilon_{i-k}(x_1) \rhd (\upsilon_i(x_0 \supset x_1) \rhd x_2))$	$(k \in i \in (n \setminus 1), \mu(i) = \mu(n-1-k) = 0 \neq \mu(i-k))$
$\upsilon_{n-1-k} \rhd (\upsilon_i(x_0 \supset x_1) \rhd \upsilon_{i-k}(x_1))$	$(n \neq 2, k \in i \in (n \setminus 1), \mu(i) = \mu(n-1-k) = 0 = \mu(i-k))$
$\upsilon_{n-1-k} \rhd (\upsilon_{i-k}(x_1) \rhd \upsilon_i(x_0 \supset x_1))$	$(k \in i \in (n \setminus 1), \mu(i) \neq \mu(n-1-k) = 0 \neq \mu(i-k))$
$\upsilon_{i-k}(x_1) \rhd (\upsilon_i(x_0 \supset x_1) \rhd \upsilon_{n-1-k})$	$(k \in i \in (n \setminus 1), \mu(i) = 0 \neq \mu(n-1-k) = \mu(i-k))$
$(\upsilon_{n-1-k} \veebar_{\rhd} \upsilon_{i-k}(x_1)) \veebar_{\rhd} \upsilon_i(x_0 \supset x_1)$	$(k \in i \in (n \setminus 1), \mu(i) = \mu(n-1-k) = 1 \neq \mu(i-k))$

$(\upsilon_{n-1-k} \rhd x_2) \rhd ((\upsilon_{i-k}(x_1) \rhd x_2) \rhd$
$(\upsilon_i(x_0 \supset x_1) \rhd x_2))$ $\quad (k \in i \in (n \setminus 1), \mu(i) = 0 = \mu(i-k) \neq \mu(n-1-k))$

$(\upsilon_{n-1-k} \rhd x_2) \rhd ((\upsilon_i(x_0 \supset x_1) \rhd x_2) \rhd$
$(\upsilon_{i-k}(x_1) \rhd x_2))$ $\quad (k \in i \in (n \setminus 1), \mu(i) = 1 = \mu(n-1-k) = \mu(i-k))$

$(\upsilon_{i-k}(x_1) \rhd x_2) \rhd ((\upsilon_i(x_0 \supset x_1) \rhd x_2) \rhd$
$(\upsilon_{n-1-k} \rhd x_2))$ $\quad (k \in i \in (n \setminus 1), \mu(i) \neq 0 = \mu(n-1-k) = \mu(i-k))$

$\upsilon_{n-k} \rhd (\upsilon_{i-k}(x_1) \rhd (\upsilon_i(x_0 \supset x_1) \rhd x_2))$ $\quad (i \in (n \setminus 1), k \in (i \setminus 1), \mu(i) = \mu(n-k) = 1 \neq \mu(i-k))$

$\upsilon_{n-k} \rhd (\upsilon_{i-k}(x_1) \rhd \upsilon_i(x_0 \supset x_1))$ $\quad (i \in (n \setminus 1), k \in (i \setminus 1), \mu(i) \neq \mu(n-k) = 1 \neq \mu(i-k))$

$\upsilon_{n-k} \rhd (\upsilon_i(x_0 \supset x_1) \rhd \upsilon_{i-k}(x_1))$ $\quad (i \in (n \setminus 1), k \in (i \setminus 1), \mu(i) = \mu(n-k) = 1 = \mu(i-k))$

$\upsilon_{i-k}(x_1) \rhd (\upsilon_i(x_0 \supset x_1) \rhd \upsilon_{n-k})$ $\quad (i \in (n \setminus 1), k \in (i \setminus 1), \mu(i) \neq \mu(n-k) = 0 = \mu(i-k))$

$(\upsilon_{n-k} \veebar_{\rhd} \upsilon_{i-k}(x_1)) \veebar_{\rhd} \upsilon_i(x_0 \supset x_1)$ $\quad (i \in (n \setminus 1), k \in (i \setminus 1), \mu(i) = \mu(n-k) = 0 \neq \mu(i-k))$

$(\upsilon_{n-k} \rhd x_2) \rhd ((\upsilon_{i-k}(x_1) \rhd x_2) \rhd$
$(\upsilon_i(x_0 \supset x_1) \rhd x_2))$ $\quad (i \in (n \setminus 1), k \in (i \setminus 1), \mu(i) \neq \mu(n-k) = 0 \neq \mu(i-k))$

$(\upsilon_{n-k} \rhd x_2) \rhd ((\upsilon_i(x_0 \supset x_1) \rhd x_2) \rhd$
$(\upsilon_{i-k}(x_1) \rhd x_2))$ $\quad (i \in (n \setminus 1), k \in (i \setminus 1), \mu(i) = \mu(n-k) = 0 = \mu(i-k))$

$(\upsilon_{i-k}(x_1) \rhd x_2) \rhd ((\upsilon_i(x_0 \supset x_1) \rhd x_2) \rhd$
$(\upsilon_{n-k} \rhd x_2))$ $\quad (i \in (n \setminus 1), k \in (i \setminus 1), \mu(i) \neq \mu(n-k) = 1 = \mu(i-k))$

$\upsilon_{n-i} \rhd \upsilon_i(x_0 \supset x_1)$ $\quad (i \in N_0 \not\ni (n-i))$
$\upsilon_i(x_0 \supset x_1) \rhd \upsilon_{n-i}$ $\quad (i \in N_1 \not\ni (n-i))$
$\upsilon_{n-i} \rhd (\upsilon_i(x_0 \supset x_1) \rhd x_2)$ $\quad (i \in N_1 \ni (n-i))$
$\upsilon_{n-i} \veebar_{\rhd} \upsilon_i(x_0 \supset x_1)$ $\quad (n \neq 2, i \in N_0 \ni (n-i))$
$\upsilon_i(x_1) \rhd \upsilon_i(x_0 \supset x_1)$ $\quad (n \neq 2, i \in N_0)$
$\upsilon_i(x_0 \supset x_1) \rhd \upsilon_i(x_1)$ $\quad (i \in N_1)$

It is remarkable that, in the classical case, when $n = 2$, the additional axioms of $\mathcal{L}_n$ are exactly the Excluded Middle Law axiom $(x_0 \veebar_{\rhd} \neg x_0) = ((x_0 \supset \neg x_0) \supset \neg x_0)$ and the Ex Contradictione Quodlibet axiom $x_0 \supset (\neg x_0 \supset x_1)$, $\mathcal{L}_2$ being a well-known natural Hilbert-style axiomatization of the classical logic. And what is more, $\mathcal{L}_n$ grows just polynomially (more precisely, quadratically) on n, so it eventually looks relatively good, the additional axioms of $\mathcal{L}_3$ being as follows, where $i \in 2$:

$$\neg x_0 \rhd (x_0 \rhd x_1) \qquad \neg^i x_i \rhd ((x_0 \supset x_1) \rhd \neg^i x_{1-i}) \qquad \neg x_0 \rhd (x_0 \supset x_1)$$
$$x_0 \rhd \neg\neg x_0 \qquad x_0 \rhd (\neg x_1 \rhd \neg(x_0 \supset x_1)) \qquad x_1 \rhd (x_0 \supset x_1)$$
$$\neg\neg x_0 \rhd x_0 \qquad (x_0 \veebar_{\rhd} \neg x_1) \veebar_{\rhd} (x_0 \supset x_1) \qquad \neg(\neg x_0 \supset x_1) \rhd \neg x_1$$

Concluding this discussion, we should like to highlight that, though, in general, an analytical expression (if any, at all) for $\bar{\upsilon}$ has not been known yet, the constructive proof of Proposition 6.10 of Pynko (2009) has been implemented upon the basis of SCWI-Prolog resulting in a quite effective logical program (taking less than second up to $n = 1000$) calculating $\bar{\upsilon}$, and so immediately yielding definitive explicit formulations of both $\mathcal{T}$ (in particular, of the Gentzen-style axiomatization $\mathcal{S}^{(0,0)}_{\mathcal{A},\mathcal{T}}$ of Ł$_n$; cf. Pynko (2004)) and the Hilbert-style axiomatization $\mathcal{L}_n$ of Ł$_n$ found above.

6.4. Hałkowska-Zajac logic

Here, it is supposed that $\Sigma \triangleq \Sigma_{\sim,+}$, $(\mathfrak{A}{\restriction}\Sigma_+) \triangleq \mathfrak{D}_3$, $\sim^{\mathfrak{A}} i \triangleq (\min(1,i) \cdot (3-i))$, for all $i \in 3$, and $D^{\mathcal{A}} \triangleq \{0,2\}$, in which case $\mathcal{A}$, defining the logic HZ Hałkowska and Zajac (1988), is $\supset$-implicative, where $(x_0 \supset x_1) \triangleq ((\sim x_0 \wedge \sim x_1) \vee x_1)$ is secondary, while $\{x_0, \sim x_0\}$ is an equality determinant for $\mathcal{A}$ (cf. Example 2 of Pynko (2004)), and so $\mathcal{A}'' = \varnothing$ and $\mathcal{A}''' = \{\vdash \sim x_0, x_0\}$. First, we have $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}} a = a$, for all $a \in A$. Therefore, one can take $\lambda_{\mathcal{T}}(\sim x_0, \sim) = \{x_0 \vdash\}$ and $\rho_{\mathcal{T}}(\sim x_0, \sim) = \{\vdash x_0\}$ to satisfy (5.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(\sim x_0, \sim) = \{\sim\sim x_0 \vdash x_0\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(\sim x_0, \sim) = \{x_0 \vdash \sim\sim x_0\}$. Next, consider any $a, b \in A$. Then, $\sim^{\mathfrak{A}}(a(\wedge/\vee)^{\mathfrak{A}} b) \in D^{\mathcal{A}}$ iff either/both $\sim^{\mathfrak{A}} a \in D^{\mathcal{A}}$ or/and $\sim^{\mathfrak{A}} b \in D^{\mathcal{A}}$. Therefore, one can take $\lambda_{\mathcal{T}}(\sim x_0, \vee) = \{\sim x_0, \sim x_1 \vdash\}$ and $\rho_{\mathcal{T}}(\sim x_0, \vee) = \{\vdash \sim x_0; \vdash \sim x_1\}$ to satisfy (5.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(\sim x_0, \vee) = \{\sim(x_0 \vee x_1) \vdash \sim x_0; \sim(x_0 \vee x_1) \vdash \sim x_1\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(\sim x_0, \vee) = \{\sim x_0, \sim x_1 \vdash \sim(x_0 \vee x_1)\}$. Likewise, one can take $\lambda_{\mathcal{T}}(\sim x_0, \wedge) = \{\sim x_0 \vdash; \sim x_1 \vdash\}$ and $\rho_{\mathcal{T}}(\sim x_0, \wedge) = \{\vdash \sim x_0, \sim x_1\}$ to satisfy (5.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(\sim x_0, \wedge) = \{\sim(x_0 \wedge x_1) \vdash \sim x_0, \sim x_1\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(\sim x_0, \wedge) = \{\sim x_0 \vdash \sim(x_0 \wedge x_1); \sim x_1 \vdash \sim(x_0 \wedge x_1)\}$. Moreover, $(a(\wedge/\vee)^{\mathfrak{A}} b) \in D^{\mathcal{A}}$ iff both $(a = 1) \Rightarrow (b = (0/2))$ and $(b = 1) \Rightarrow (a = (0/2))$. Therefore, one can take $\rho_{\mathcal{T}}(x_0, \wedge) = \{\vdash x_0, x_1; \vdash \sim x_0, x_1; \vdash \sim x_1, x_0\}$ and $\lambda_{\mathcal{T}}(x_0, \wedge) = \{x_0, x_1 \vdash; x_0, \sim x_0 \vdash; x_1, \sim x_1 \vdash\}$ to satisfy (5.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(x_0, \wedge) = \{(x_0 \wedge x_1) \vdash x_0, x_1; (x_0 \wedge x_1) \vdash \sim x_0, x_1; (x_0 \wedge x_1) \vdash \sim x_1, x_0\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(x_0, \wedge) = \{x_0, x_1 \vdash (x_0 \wedge x_1); x_0, \sim x_0 \vdash (x_0 \wedge x_1); x_1, \sim x_1 \vdash (x_0 \wedge x_1)\}$. Likewise, one can take $\rho_{\mathcal{T}}(x_0, \vee) = \{\vdash x_0, x_1; \sim x_1 \vdash x_0; \sim x_0 \vdash x_1\}$ and $\lambda_{\mathcal{T}}(x_0, \vee) = \{x_0, x_1 \vdash; \vdash \sim x_0; \vdash \sim x_1\}$ to satisfy (5.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(x_0, \vee) = \{(x_0 \vee x_1) \vdash x_0, x_1; \sim x_1, (x_0 \vee x_1) \vdash x_0; \sim x_0, (x_0 \vee x_1) \vdash x_1\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(x_0, \vee) = \{x_0, x_1 \vdash (x_0 \vee x_1); \vdash \sim x_0, (x_0 \vee x_1); \vdash \sim x_1, (x_0 \vee x_1)\}$. In this way, taking Corollary 3.8(ii) into account, we eventually get:

Corollary 6.20. *HZ is axiomatized by the calculus $\mathcal{HZ}$ resulted from $\mathfrak{I}^{\mathrm{PL}}_{\supset}$ by adding the axioms* (6.1), (6.2), (6.3) *and the following ones, where $i \in 2$:*

$$(x_0 \supset x_2) \supset ((x_1 \supset x_2) \supset ((x_0 \wedge x_1) \supset x_2)) \qquad x_0 \supset (x_1 \supset (x_0 \wedge x_1))$$
$$(\sim x_i \supset x_2) \supset ((x_{1-i} \supset x_2) \supset ((x_0 \wedge x_1) \supset x_2)) \qquad x_i \supset (\sim x_i \supset (x_0 \wedge x_1))$$
$$(x_0 \supset x_2) \supset ((x_1 \supset x_2) \supset ((x_0 \vee x_1) \supset x_2)) \qquad x_0 \supset (x_1 \supset (x_0 \vee x_1))$$
$$(\sim x_i \supset (x_0 \vee x_1)) \supset (x_0 \vee x_1) \qquad \sim x_{1-i} \supset ((x_0 \vee x_1) \supset x_i)$$
$$(\sim x_0 \supset x_0) \supset x_0$$

In this connection, recall that an *infinite* Hilbert-style axiomatization of HZ has been due to Zbrzezny (1990).

7. Conclusions

As a matter of fact, Subsection 6.2 has provided finite Hilbert-style axiomatizations of *all* miscellaneous expansions of B_4 studied in Pynko (1999), even though such is not considered *explicitly*. More precisely, an expansion of such a kind with[out] implication is axiomatized by the union of appropriate calculi presented in Subsubsection[s] 6.2.3 [resp., 6.2.1 and 6.2.2].

Even though Section 6 does not exhaust *all* interesting applications of Section 5, it has definitely incorporated *most acute* ones.

In general, the effective nature of the present elaboration definitely makes the paper a part of *Applied* Non-Classical Logic, especially due to quite effective program implementations invented in this connection.

References

Asenjo, F. G., & Tamburino, J. (1975). Logic of antinomies. *Notre Dame Journal of Formal Logic*, *16*, 272–278.

Belnap, N. D., Jr. (1977). A useful four-valued logic. In J. M. Dunn & G. Epstein (Eds.), *Modern uses of multiple-valued logic* (pp. 8–37). Dordrecht: D. Reidel Publishing Company.

D'Ottaviano, I. M. L., & Epstein, R. L. (1988). A paraconsistent many-valued propositional logic: **J3**. *Reports on Mathematical Logic*, *22*, 89–103.

Dyrda, K., & Prucnal, T. (1980). On finitely based consequence determined by a distributive lattice. *Bulletin of the Section of Logic*, *9*, 60–66.

Hałkowska, K., & Zajac, A. (1988). O pewnym, trójwartościowym systemie rachunku zdań. *Acta Universitatis Wratislaviensis. Prace Filozoficzne*, *57*, 41–49.

Kleene, S. C. (1952). *Introduction to metamathematics*. New York: D. Van Nostrand Company.

Łoś, J., & Suszko, R. (1958). Remarks on sentential logics. *Indagationes Mathematicae*, *20*, 177–183.

Łukasiewicz, J. (1920). O logice trójwartościowej. *Ruch Filozoficzny*, *5*, 170–171.

Mendelson, E. (1979). *Introduction to mathematical logic* (2nd ed.). New York: D. Van Nostrand Company.

Peirce, C. (1885). On the Algebra of Logic: A Contribution to the Philosophy of Notation. *American Journal of Mathematics*, *7*, 180-202.

Popov, V. M. (1989). Sequential formulations of some paraconsistent logical systems. In V. A. Smirnov (Ed.), *Syntactic and semantic investigations of non-extensional logics* (pp. 285–289). Moscow: Nauka. (In Russian)

Priest, G. (1979). The logic of paradox. *Journal of Philosophical Logic*, *8*, 219–241.

Pynko, A. P. (1995). Characterizing Belnap's logic via De Morgan's laws. *Mathematical Logic Quarterly*, *41*(4), 442–454.

Pynko, A. P. (1999). Functional completeness and axiomatizability within Belnap's four-valued logic and its expansions. *Journal of Applied Non-Classical Logics*, *9*(1/2), 61–105. (Special Issue on Multi-Valued Logics)

Pynko, A. P. (2004). Sequential calculi for many-valued logics with equality determinant. *Bulletin of the Section of Logic*, *33*(1), 23–32.

Pynko, A. P. (2009). Distributive-lattice semantics of sequent calculi with structural rules. *Logica Universalis*, *3*(1), 59–94.

Pynko, A. P. (2014). Minimal sequent calculi for monotonic chain finitely-valued logics. *Bulletin of the Section of Logic*, *43*(1/2), 99–112.

Pynko, A. P. (2015). Minimal Sequent Calculi for łukasiewicz's Finitely-Valued Logics. *Bulletin of the Section of Logic*, *44*(3/4), 149–153.

Zbrzezny, A. (1990). The Hilbert-type axiomatization of some three-valued propositional logic. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, *36*, 415–421.