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Evaluating the $M|D|\infty$ Queue Busy Cycle Distribution

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Abstract

Given the busy period major importance in queuing systems, it is also relevant to study the busy cycle. In this work some interesting results on $M|G|\infty$ queue system busy cycle distribution are presented. They are emphasized for the M|D|∞ queue system and a numerical method to compute the M|D|∞ queue system busy cycle distribution function is presented.

Keywords: M|D|∞, M|G|∞, busy cycle, distribution function.

1 Introduction

A queue system busy period is a period that begins when a customer arrives at the system finding it empty, ends when a customer abandons the system letting it empty and, throughout its progress, there is always at least one customer present. An idle period followed by a busy period is a busy cycle.

In the M|G|∞ queue system the customers arrive according to a Poisson process at rate λ, receive a service which time length is a positive random variable with distribution function $G(.)$ and mean α and, when they arrive, each one finds immediately an available server. Each customer service is independent from the other customers' services and from the arrivals process. The traffic intensity is $\rho = \lambda\alpha$.

Call I, B and Z the time length random variable of the idle period, the busy period and the busy cycle respectively; $i(t), b(t)$ and $z(t)$ are the correspondent probability density functions and $I(t), B(t)$ and $Z(t)$ the distribution functions.

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2 General Results

Evidently $Z = I + B$ and being I and B independent, see [2], the distribution of Z is the I and B distributions convolution. Then, being $\bar{Z}(s), \bar{I}(s)$ and $\bar{B}(s)$ the Z, B and I, respectively, Laplace transforms

$$\bar{Z}(s) = \bar{I}(s)\,\bar{B}(s) \qquad (2.1)$$

where

$$\bar{I}(s) = \frac{\lambda}{\lambda + s} \qquad (2.2)$$

as it happens for any queue with Poisson arrivals process and

$$\bar{B}(s) = 1 + \lambda^{-1}\left(s - \frac{1}{\int_0^\infty e^{-st-\lambda\int_0^t [1-G(v)]dv} dt}\right) \qquad (2.3),$$

see again [2].

Consequently

$$E[Z^n] = \sum_{p=0}^{\infty} \binom{n}{p} \frac{p!}{\lambda^p} E[B^{n-p}], n = 1,2,\dots \qquad (2.4)$$

where, see [6],

$$E[B^n] = (-1)^{n+1}\left\{\frac{e^\rho}{\lambda} n\, C^{(n-1)}(0) - e^\rho \sum_{p=1}^{n-1} (-1)^{n-p} \binom{n}{p} E[B^{n-p}] C^{(p)}(0)\right\},$$

$$n = 1,2,\dots \quad (2.5)$$

and

$$C^{(n)}(0) = \int_0^\infty (-t)^n e^{-\lambda\int_0^t [1-G(v)]dv} \lambda\big(1 - G(t)\big) dt, n = 01,2,\dots \quad (2.6).$$

So

$$E[Z] = \frac{e^\rho}{\lambda} \qquad (2.7)$$

does not depend on the service time distribution form, except for its mean[2]. But $E[Z^n], n \geq 2$ depend on the whole service time distribution structure.

For the M|D|∞ queue system – constant[3]service times with value α -

$$\bar{B}(s) = 1 + \lambda^{-1}\left(s - \frac{(s+\lambda)s}{\lambda e^{-(s+\lambda)\alpha} + s}\right) \qquad (2.8)$$

obtaining, by Laplace transform inversion, see [3][4],

$$b(t) = \sum_{n=0}^{\infty}\left(\frac{d}{dt}\frac{c(t)}{e^{-\rho}}\right) * \left(\frac{d}{dt}\frac{1-d(t)}{1-e^{-\rho}}\right)^{*n} e^{-\rho}(1-e^{-\rho})^n \qquad (2.9)$$

where $\frac{c(t)}{e^{-\rho}} = \begin{cases} 0, t < \alpha \\ 1, t \geq \alpha \end{cases} = G(t)$ and $\frac{1-d(t)}{1-e^{-\rho}} = \begin{cases} \frac{1-e^{-\lambda t}}{1-e^{-\rho}}, t < \alpha \\ 1, t \geq \alpha \end{cases}$.

Then

$$\bar{Z}(s) = 1 - \frac{s}{\lambda e^{-(s+\lambda)\alpha} + s} \qquad (2.10)$$

and

$$z(t) = \left(\lambda e^{-\lambda t}\right) * b(t), t \geq 0 \qquad (2.11).$$

Still

$$\begin{gathered} C^{(0)}(0) = 1 - e^{-\rho} \\ C^{(n)}(0) = -e^{-\rho}(-\alpha)^n - \frac{n}{\lambda}\, C^{(n-1)}(0),\ n = 1,2,\ldots \end{gathered} \qquad (2.12),$$

see again [6].

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[2] In these circumstances it is usual to say that it is insensible to the service time distribution.

[3] That is: **D**eterministic service times.

[4] $*$ is the convolution operator.

3 The $M|D|\infty$ Queue Busy Cycle Distribution Function

The expression (2.11) for $z(t)$, in the former section, allows the busy cycle distribution structure interpretation for the $M|D|\infty$ queue. But it fails in the task of presenting an easy expression for the distribution function $Z(t)$ computation.

This may be done, for example, with an algorithm created by Platzman, Ammons and Bartholdi III, see [1] [5], that allows the distribution functions computation since the correspondent Laplace transform in round form is known, as it is now the case, remember (2.10). Unhappily the same does not happen for other $M|G|\infty$ systems what inhibits the use of this algorithm.

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[5] It is generally said that an algorithm is "accurate" if it looks for solving a problem "close" to the one that is supposed to solve. An algorithm is "precise" if it gets a solution "close" to the one of the problem that it is trying to solve. .More concretely, being Δt $(\Delta t > 0)$ the accuracy and Δp $\left(0 < \Delta p < \frac{1}{2}\right)$ the precision required, the approximation τ of $P[X > t]$ must satisfy the condition

$$P[X \geq t + \Delta t] - \Delta p \leq \tau \leq P[X > t - \Delta t] + \Delta p \qquad (3.1).$$

Platzman, Ammons and Bartholdi III suggest doing

$$\tau = \frac{U-t+\Delta t}{U-L+2\Delta t} + \sum_{n=1}^{N} \frac{\alpha^{n^2}}{\pi n} im\{(\beta^n - \gamma^n)L(j\omega n)\} \qquad (3.2)$$

where $K = \log\frac{2}{\Delta p}$, D=$\frac{\Delta t}{\sqrt{2K}}$, $\omega = \frac{2\pi}{U-L+2\Delta t}$, N=$\left[\frac{2K}{\omega\Delta t}\right]$, being $[\cdot]$ the characteristic of a real number, $\alpha = e^{-D^2\frac{\omega^2}{2}}$, $\beta = e^{j(U+\Delta t)\omega}$, $\gamma = e^{jt\omega}$, U and L are numbers such that $1 - P[L \leq X \leq U] \ll \Delta p, j = \sqrt{-1}$ and $im(\cdot)$ designates the imaginary part of a complex number. $L(j\omega n)$ is the Laplace transform value in $j\omega n$. They demonstrate that the approximation so defined fulfills the condition (3.1).

The algorithm implementation, for details see [4], is computationally performed through a FORTRAN program, see [5], and the results of some experiences are presented in the Annex.

The values of α, λ, Δt and Δp must be specified and the values of t for which the values of $Z(t)$, called $Z^c(t)$, are wanted.

As for the goodness of the obtained results, it is tested computing the errors of $E[Z^c]$ and $VAR\ [Z^c]$, computed after them, in relation with the true values of $E[Z]$ and $VAR[Z]$ that are available for this queue system. The exception is the first experience where, with α=0, the situation is the one of a pure Poisson process. So, the results obtained (2nd column in Table 1) are compared with the Poisson process ones (3rd column in Table 1). Generally, the Z^c values fit well.

References

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ANNEX

Table 1. Experience 1: α=0, λ=1, Δ t= 0.01 and Δp=0,001

t	Z^c(t)	Poisson Process
0	0.00020928263	0.000…
0.5	0.39354845	0.39346934
1	0.63201874	0.632120559
1.5	0.77676630	0.77686984
2	0.86456292	0.864664717
2.5	0.91781115	0.917915001
3	0.95011103	0.95021212932
3.5	0.96969878	0.969802617

Table 2. Experience 2: α=1, λ=1, Δt=0.01 and Δp=0,001

t	Z^c(t)	t	Z^c(t)
0.5	0.00070788896	4.5	0.89332950
1	0.00078194999	5	0.92884773
1.5	0.18467983	5.5	0.95303684
2	0.36851909	6	0.96932029
2.5	0.53561949	6.5	0.98016983
3	0.66881525	7	0.98734205
3.5	0.76919734	7.5	0.99205017
4	0.84198290		
$E[Z] = 2.718281829$ $E[Z^c] = 2.605018789$ $\varepsilon = 4\%$		$VAR[Z] = 1.9444392442$ $VAR[Z^c] = 1.875647136$ $\varepsilon = 3.5\%$	

Table 3. Experience 3: α=1, λ=2, Δt=0.01 and Δp=0,001

t	Z^c(t)	t	Z^c(t)
0.5	0.00038790601	7.5	0.92047894
1	0.00045109048	8	0.93518191
1.5	0.13572108	8.5	0.94718128
2	0.27099844	9	0.95697385
2.5	0.39718168	9.5	0.96496373
3	0.50513958	10	0.97148519
3.5	0.59509700	10.5	0.97680729
4	0.66922503	11	0.98115152
4.5	0.72997826	11.5	0.96469930
5	0.77964925	12	0.98759257
5.5	0.82022225	12.5	0.98995178
6	0.85335999	13	0.99188309
6.5	0.88039940	13.5	0.99344980
7	0.92047130	14	0.99473917

$E[Z] = 3.69452805$ $E[Z^c] = 3.606224458$ $\varepsilon = 2.4\%$	$VAR[Z] = 6.260481408$ $VAR[Z^c] = 5.358674148$ $\varepsilon = 14\%$

Table 4. Experience 1: α=2, λ=1, Δt=0.01 and Δp=0,01

t	Z^c(t)	t	Z^c(t)
0.5	0.00039526703	14	0.90255320
1	0.00039531649	14.5	0.91201680
1.5	0.00039744257	15	0.92056465
2	0.00042999497	15.5	0.92828899
2.5	0.0068082088	16	0.93526571
3	0.13566480	16.5	0.94157290
3.5	0.20333376	17	0.94726365
4	0.27105104	17.5	0.95241045
4.5	0.33643096	18	0.95705801
5	0.39722785	18.5	0.96125179
5.5	0.45344632	19	0.96504825
6	0.50523263	19.5	0.96847575
6.5	0.55233818	20	0.97157025
7	0.59518069	20.5	0.97437018
7.5	0.63407224	21	0.97689431
8	0.66930794	21.5	0.97917509
8.5	0.70120662	22	0.98124003
9	0.73005634	22.5	0.98309797
9.5	0.75615197	23	0.98477888
10	0.77973318	23.5	0.98630297
10.5	0.80105113	24	0.98767584
11	0.82031202	24.5	0.98891764
11.5	0.83771467	25	0.99003869
12	0.85343867	25.5	0.99104917
12.5	0.86764937	26	0.99196279
13	0.88047999	26.5	0.99279278
13.5	0.89207541	27	0.99353820
$E[Z] = 7.389056099$ $E[Z^c] = 7.200722486$ $\varepsilon = 2.5\%$		$VAR[Z] = 25.04192563$ $VAR[Z^c] = 20.69584719$ $\varepsilon = 17\%$	