The Polyhedral Structure and Complexity of Multistage Stochastic Linear Problem with General Cost Distribution

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Abstract

By studying the intrinsic polyhedral structure of multistage stochastic linear problems (MSLP), we show that a MSLP with an arbitrary cost distribution is equivalent to a MSLP on a finite scenario tree. More precisely, we show that the expected cost-to-go function, at a given stage, is affine on each cell of a chamber complex i.e., on the common refinement of the complexes obtained by projecting the faces of a polyhedron. This chamber complex is independent of the cost distribution. Furthermore, we examine several important special cases of random cost distributions, exponential on a polyhedral cone, or uniform on a polytope, and obtain an explicit description of the supporting hyperplanes of the cost-to-go function, in terms of certain valuations attached to the cones of a normal fan. This leads to fixed-parameter tractability results, showing that MSLP can be solved in polynomial time when the number of stages together with certain characteristic dimensions are fixed.

1 Introduction

Stochastic programming is a powerful modeling paradigm for optimization under uncertainty that has found many applications in energy, logistics or finance (see e.g. [WZ05]). Linear stochastic programs constitute an important class of stochastic programs. They have been thoroughly studied, see e.g. [BL11, Pré13]. One reason for this interest is the availability of efficient linear solvers and the use of dedicated algorithms leveraging the special structure of linear stochastic programs ([VSW69, Bir85]).

In this paper, we study the polyhedral structure of cost-to-go functions of MSLP. This leads to explicit representations of these functions and to new complexity results.

1.1 Polyhedrality of cost-to-go function

We consider the multistage stochastic linear program (MSLP), in which several decisions are taken, based on successive observations of random events.

Given a sequence of independent random variables $\boldsymbol{c}_t$ and $\boldsymbol{\xi}_t = (\boldsymbol{T}_t, \boldsymbol{W}_t, \boldsymbol{h}_t)$, indexed by $t \in [t_{\max}] := \{1, \dots, t_{\max}\}$, we define the *cost-to-go* function V_t inductively as follows. We set $V_{t_{\max}+1} \equiv$

0 and for all $t \in [t_{\max}]$:

$$\begin{aligned} V_t(x) &:= \mathbb{E}\big[\hat{V}_t(x, \boldsymbol{c}_t, \boldsymbol{\xi}_t)\big] \\ \hat{V}_t(x_{t-1}, c_t, \xi_t) &:= \min_{x_t \in \mathbb{R}^{n_t}} \quad c_t^\top x_t + V_{t+1}(x_t) \\ &\qquad\quad \text{s.t.} \quad T_t x_{t-1} + W_t x_t \leqslant h_t \end{aligned}$$

where $x_{t-1} \in \mathbb{R}^{n_{t-1}}$, $c_t \in \mathbb{R}^{n_t}$ and $\xi_t = (T_t, W_t, h_t) \in \mathbb{R}^{q_t \times n_{t-1}} \times \mathbb{R}^{q_t \times n_t} \times \mathbb{R}^{q_t}$.

The multistage stochastic linear problem (MSLP) is the minimisation problem (1) specialized to $t = 1$, with value $\hat{V}_1(x_0, c_1, \xi_1)$, where $x_0 \in \mathbb{R}^{n_0}$, $c_1 \in \mathbb{R}^{n_1}$, $\xi_1 \in \mathbb{R}^{q_1 \times n_0} \times \mathbb{R}^{q_1 \times n_1} \times \mathbb{R}^{q_1}$ are given.

In this paper, we choose to distinguish the random cost $\boldsymbol{c}$ from the noise $\boldsymbol{\xi}$ affecting the constraints. Indeed our results require $\boldsymbol{\xi}$ to be finitely supported (see Examples 1 and 2) while $\boldsymbol{c}$ can have a continuous distribution. This separation does not preclude correlation between $\boldsymbol{c}_t$ and $\boldsymbol{\xi}_t$. However, we require $\{(\boldsymbol{c}_t, \boldsymbol{\xi}_t)\}_{t \in [t_{\max}]}$ to be a sequence of independent random variables to leverage Dynamic Programming, even though some results can be extended to dependent $(\boldsymbol{\xi}_t)_{t \in [t_{\max}]}$.

In order to solve the MSLP, the irreducible difficulty is perhaps the study and the representation of the cost-to-go function V, defined as the expectation of the optimal cost of a linear program with random data, and affine constraints in the variable x:

$$V(x) = \mathbb{E}\begin{bmatrix} \min\limits_{y \in \mathbb{R}^m} & \boldsymbol{c}^\top y \\ \text{s.t.} & \boldsymbol{T}x + \boldsymbol{W}y \leqslant \boldsymbol{h} \end{bmatrix} . \tag{2}$$

The cost vector $\boldsymbol{c} \in \mathbb{R}^m$, the constraint matrices $\boldsymbol{T} \in \mathbb{R}^{q \times n}$ and $\boldsymbol{W} \in \mathbb{R}^{q \times m}$ and the vector $\boldsymbol{h} \in \mathbb{R}^q$ are random variables.

If $(\boldsymbol{c}, \boldsymbol{\xi})$ have a finite support, it is known that V is *polyhedral*, meaning that it takes value in $\mathbb{R} \cup \{+\infty\}$ and its epigraph is a (possibly empty) polyhedron. Indeed, for each $(c, \xi) \in \operatorname{supp}(\boldsymbol{c}, \boldsymbol{\xi})$, $Q^{c,\xi} : (x, y) \to \ c^\top y + \mathbb{I}_{Tx + Wy \leqslant h}$ is polyhedral. Thus, $\hat{V}(\cdot, c, \xi) = \min_{y \in \mathbb{R}^m} Q^{c,\xi}(\cdot, y)$ is polyhedral as $\operatorname{epi} \hat{V}(\cdot, c, \xi)$ is a projection of $\operatorname{epi} Q^{c,\xi}$ (see [JKM08]). Finally, V, being a positive linear combination of polyhedral functions, is also polyhedral.

Our results show that V is polyhedral without any finite support condition on the distribution of $\boldsymbol{c}$. More precisely, we show that we can replace $\boldsymbol{c}$ by a finitely supported $\check{\boldsymbol{c}}$ that yields the same expected cost-to-go function, V. Moreover, there exists a finite polyhedral partition of the space that does not depends on the distribution of $\boldsymbol{c}$, such that V is affine on each of its elements.

As shown by the following examples, this theorem is tight: if $\boldsymbol{T}$, $\boldsymbol{W}$ or $\boldsymbol{h}$ have an arbitrary distribution, V may not be polyhedral. Let $\boldsymbol{u}$ be a uniform random variable on $[0, 1]$.

Example 1 (Stochastic $\boldsymbol{h}$). If $\boldsymbol{T} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\boldsymbol{W} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$, $\boldsymbol{c} = 1$ and $\boldsymbol{h} = \begin{pmatrix} -\boldsymbol{u} \\ 0 \end{pmatrix}$, then

$$V(x) = \mathbb{E}\begin{bmatrix} \min\limits_{y \in \mathbb{R}^m} & y \\ \text{s.t.} & \boldsymbol{u} \leqslant y \\ & x \leqslant y \end{bmatrix} = \mathbb{E}\big[\max(x, \boldsymbol{u})\big] = \begin{cases} \frac{1}{2} & \text{if } x \leqslant 0 \\ \frac{x^2+1}{2} & \text{if } x \in [0, 1] \\ x & \text{if } x \geqslant 1 \ . \end{cases}$$

Example 2 (Stochastic $\boldsymbol{T}$). If $\boldsymbol{T} = \begin{pmatrix} \boldsymbol{u} \\ 0 \end{pmatrix}$, $\boldsymbol{W} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$, $\boldsymbol{c} = 1$ and $\boldsymbol{h} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, then

$$V(x) = \mathbb{E}\begin{bmatrix} \min\limits_{y \in \mathbb{R}^m} & y \\ \text{s.t.} & \boldsymbol{u}x \leqslant y \\ & 1 \leqslant y \end{bmatrix} = \mathbb{E}\big[\max(\boldsymbol{u}x, 1)\big] = \begin{cases} 1 & \text{if } x \leqslant 1 \\ \frac{x}{2} + \frac{1}{2x} & \text{if } x \geqslant 1 \end{cases}$$

1.2 Contribution and literature review

Most results for MSLP with continuous distributions rely on discretizing the distributions. The Sample Average Approximation (SAA) method (see e.g. [SDR14, Chap. 5]) samples the costs and constraints. It relies on probabilistic results based on a uniform law of large number to give statistical guarantees. Obtaining a good approximation requires a large number of scenarios. In order to alleviate the computations, we can use scenario reduction techniques (see [DGKR03, HR03]). Latin Hypercube Sampling and variance reduction methods are also used to produce scenarios. Finally one generate heuristically "good" scenarios, representing the underlying distribution (see [KW07]). Alternatively, we can leverage the structure of the problem to produce finite scenario trees (see [Kuh06, MAB14, MP18]) that yields bounds for the value of the true optimization problem.

In each of these approaches, one solves an approximate version of the stochastic program, with or without statistical guarantee. In contrast, our approach aims at solving exactly the original problem.

We rely on a geometric approach, which enlightens the polyhedral structure of MSLP. In particular, Theorem 11 gives an explicit representation of the expected cost-to-go value starting from a given point, as a sum over all the cones of the normal fan of a polyhedron defined by the constraints. The probability distribution of the cost determines a valuation associated to each of these cones. We deduce that the MSLP is equivalent to a problem with finitely many scenarios, that can be characterized through common refinement of certain polyhedral complexes, see Theorem 17. The "master formula" of Theorem 14 shows that the expected cost-to-go function is piecewise affine, and that it is affine on every cell of a specific *chamber complex*. A chamber complex [BS92, RZ96] is a polyhedral complex defined as the common refinement of the projections of faces of a polyhedron. Theorem 14 yields an explicit formula for the expected cost-to-go function, using conic coordinates and active constraints. We refer to [Zie12, Grü13, Fuk16]for background on polyhedral complexes, fans.

In order to evaluate the "master formula" of Theorem 14, we need to compute certain valuations associated to the cones of a polyhedral fan (see Lemma 13). Leveraging adapted triangulations, computing these valuations is reduced to computing expectation over simplices. In particular, when the costs have exponential distributions, we can rely on Brion's formula [Bri88] for the exponential valuation of polyhedra. When the cost has a uniform measure supported by a polyhedron, we also derive an explicit formula, involving volumes and centroids. In both cases, when the coefficients of the inputs are rational, the epigraph of the cost-to-go function is a rational polyhedron, meaning that the defining halfspaces have rational coefficients. Other remarkable cases include distributions with a special symmetry (rotational symmetry after an affine transformation), like the Gaussian distribution or the uniform distribution on an ellipsoid. Then, we are reduced to computing solid angles and spherical centroids.

This polyhedral approach leads to new complexity results. Indeed, Dyer and Stougie [DS06] proved that 2 stage stochastic programming is $\sharp P$-hard in the discrete case, by reducing the problem of graph reliability to the discrete distribution case. They stated that the computation of the volume of a polytope can be reduced to the continuous distribution case, a result which was subsequently proved in [HKW16]. Computing the volume of a polytope, as well as graph reliability, is $\sharp P$-complete. Hanasusanto, Kuhn and Wiesemann [HKW16] showed that computing an approximate solution to the 2-stage linear programming (2SLP) with continuous distribution with a sufficiently high accuracy is also $\sharp P$-hard. Other papers [SN05] studied the complexity of 2-stage linear programming 2SLP and MSLP. Most complexity results there are hardness results. In contrast, we prove that 2SLP and MSLP are fixed parameter tractable: when the dimension of the recourse space is fixed, 2SLP is polynomial-time. Moreover, when the dimensions of the first stage decision space and of the recourse space are both fixed, the whole cost-to-go function can be determined in polynomial time. Furthermore, when all the dimensions of the decision spaces are fixed, as well as the horizon (so that the number of constraints is the only free parameter), MSLP is polynomial time.

In summary, our main contributions are the following:

1. MSLP with arbitrary cost distribution and finitely supported constraints are equivalent to MSLP with discrete cost distribution;

2. the cost-to-go functions of such MSLP are polyhedral and they are affine on regions that are independent of the cost;

3. new algebraic insights on the polyhedral structure of MSLP;

4. analytical formulas for exponentially or uniformly distributed costs on a polytope;

5. fixed-parameter versions of 2SLP and MSLP are polynomial time.

The rest of the paper is laid out as follows.

In Section 2, we recall notions from the theory of polyhedra: *polyhedral complexes*, *normal fans* and *chamber complexes*. Section 3 is the workhorse of the paper, in which we leverage the polyhedral tools to construct an equivalent finitely distributed cost $\check{c}$ thus proving the polyhedrality of V. In Section 4, we show that the study of the cost-to-go function with deterministic constraints carry over to finitely supported constraints and to the multistage case. In Section 5, we show that the expectations, probabilities and conic coordinates used in the expression of V in Section 3 can be made explicit for usual distributions of $\boldsymbol{c}$. In Section 6, we apply our approach to an illustrative example. Finally, in Section 7, we draw the consequences of our results in terms of computational complexity.

1.3 Notation

As a general guideline ***bold*** letters denote random variables, normal scripts their realisation. Capital letters denote matrices or sets, calligraphic (e.g. $\mathcal{N}$) denote collections of sets. The indicator function $\mathbb{I}_{\mathsf{P}}$ (resp. $\mathbb{1}_{\mathsf{P}}$) takes value 0 (resp. 1) if P is true and $+\infty$ (resp. 0) otherwise. We set $[k] := \{1, \dots, k\}$, and we denote by $\sharp E$ the cardinal of a set E. We denote by A_I the submatrix of a matrix A, composed of the rows of indices $i \in I$. We denote by $\mathrm{Cone}(A) := A\mathbb{R}^n_+$ the cone hull of the columns of A. $x \leqslant y$ is the standard partial order, given by $\forall i, x_i \leqslant y_i$, $F \lhd G$ if F is a subface of G. $\mathcal{P} \preccurlyeq \mathcal{Q}$ if $\mathcal{P}$ is a refinement of the polyhedral complex $\mathcal{Q}$. $\operatorname{supp} \mathcal{C} := \bigcup_{C \in \mathcal{C}} E$ is the support of a collection of sets $\mathcal{C}$, $\mathcal{C}^{\max}$: the sets of maximal elements of a collection of sets $\mathcal{C}$. $\mathrm{rc}(P)$ is the recession cone of a polyhedron P. For a polyhedron P, we denote $\mathcal{F}(P)$ its faces, $\mathrm{Vert}(P)$ its vertices and $\mathrm{Ray}(P)$ a set with vectors each representing one extreme rays (for example the normalized extreme rays). P^ψ is the face of P given by $\arg\min_{x \in P} \psi^\top x$. $N_P(x)$ is the normal cone of P at x, and $\mathcal{N}(P)$ the normal fan of P. $I_{A,b}(x) := \{i \mid A_i x = b_i\}$ the set of active constraints in x for an H-representation $\{z \mid Az \leqslant b\}$, and $\mathcal{I}(A, b)$ the collection of these sets $\{I_{A,b}(x) \mid Ax \leqslant b\}$.

2 Polyhedral tools

Our proofs rely on the notions of normal fan and chamber complex of a polyhedron recalled here. These polyhedral objects reveal the geometrical structure of MSLP. Both the normal fan and the chamber complex are special polyhedral complexes.

2.1 Polyhedral complexes

Polyhedral complexes are finite collections of polyhedra satisfying some combinatorial and geometrical properties. In particular the relative interiors of the elements of a polyhedral complex (without the empty set) form a partition of their union. We refer to [DLRS10] for a complete introduction to polyhedral complexes and triangulations.

Definition 1 (Polyhedral complex)**.** *A finite collection of polyhedra $\mathcal{C}$ is a* polyhedral complex *if it satisfies i) if $P \in \mathcal{C}$ and F is a non-empty*[1] *face of P then $F \in \mathcal{C}$ and ii) if P and Q are in $\mathcal{C}$, then $P \cap Q$ is a (possibly empty) face of P.*

We denote by $\operatorname{supp} \mathcal{C} := \bigcup_{P \in \mathcal{C}} P$ *the support of a polyhedral complex. Further, if all the elements of $\mathcal{C}$ are polytopes (resp. cones, simplices, simplicial cones), we say that $\mathcal{C}$ is a* polytopal complex *(resp. a* fan*, a* simplicial complex*, a* simplicial fan*).*

We recall that a *simplex* of dimension d is the convex hull of $d+1$ affinely independent point and that a *simplicial cone* of dimension d is the conical hull of d linearly independent vectors.

Proposition 2. *For any polyhedral complex $\mathcal{C}$, the relative interiors of its elements (without the empty set) form a partition of its support:* $\operatorname{supp}(\mathcal{C}) = \bigsqcup_{P \in \mathcal{C}} \operatorname{ri}(P)$.

For example, the set of faces $\mathcal{F}(P)$ of a polyhedron P is a polyhedral complex.

Definition 3 (Refinements and triangulation)**.** *Let $\mathcal{C}$ and $\mathcal{R}$ be two polyhedral complexes, we say that $\mathcal{R}$ is a* refinement *of $\mathcal{C}$, denoted $\mathcal{R} \preccurlyeq \mathcal{C}$, if* $\operatorname{supp} \mathcal{R} = \operatorname{supp} \mathcal{C}$ *and for all cell $R \in \mathcal{R}$ there exists a cell $C \in \mathcal{C}$ containing R: $R \subset C$.*

Note that $\preccurlyeq$ define a partial order of polyhedral complexes, and the meet *associated to this order is given by the* common refinement *of two polyhedral complexes $\mathcal{C}$ and $\mathcal{C}'$ defined as the polyhedral complex of the intersections of cells of $\mathcal{C}$ and $\mathcal{C}'$:*

$$\mathcal{C} \wedge \mathcal{C}' := \{R \cap R' \mid R \in \mathcal{C}, R' \in \mathcal{C}'\}$$

A triangulation $\mathcal{T}$ of a polytope Q *is a refinement of $\mathcal{F}(Q)$ such that the cells of dimension 0 of $\mathcal{T}$ are the vertices of Q and $\mathcal{T}$ is a simplicial complex. A* triangulation $\mathcal{T}$ of a cone K *is a refinement of $\mathcal{F}(K)$ such that the cells of dimension 1 of $\mathcal{T}$ are the rays of K and $\mathcal{T}$ is a simplicial fan.*

2.2 Normal fan

The normal fan is the collection of the normal cones of all faces of a polyhedron. See [LR08] for a review of normal fan properties.

Recall that the *normal cone* of a convex set $C \subset \mathbb{R}^m$ at the point x is the set $N_C(x) := \{\alpha \in \mathbb{R}^m \mid \forall y \in C,\ \alpha^\top (y - x) \leqslant 0\}$. More generally, for a set $E \subset C$, $N_C(E) := \bigcap_{x \in E} N_C(x)$.

[1]For some authors, a polyhedral complex must contain the empty set. We do not make this requirement.

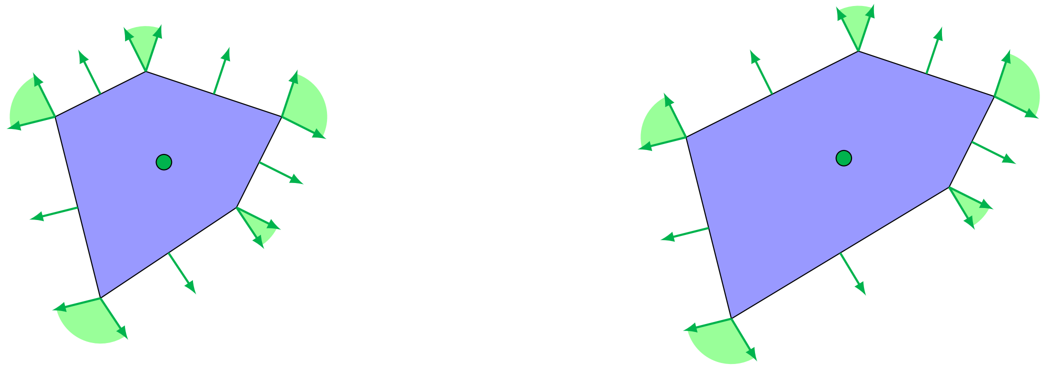


Figure 1: Two normally equivalent polytopes P and P' and their normal fan $\mathcal{N}(P) = \mathcal{N}(P')$.

Definition 4 (Normal Fan). *The* normal fan[2] *of a convex set C is the collection of polyhedral cones*

$$\mathcal{N}(C) := \{N_C(x) \mid x \in C\}$$

We say that two convex sets C and C' are normally equivalent *if they have the same normal fan : $\mathcal{N}(C) = \mathcal{N}(C')$.*

Recall that the *polar* of a convex set C is the set $C^\circ := \{\alpha \mid \forall x \in C,\ \alpha^\top x \leqslant 0\} = N_C(0)$ and the recession cone of a convex set C is given by $\mathrm{rc}(C) := \{\, r \in C \mid \forall \mu \in \mathbb{R}_+,\ \forall x \in c,\ x + \mu r \in C\}$. In particular, for a polyhedron, the recession cone and its polar are given by

$$\mathrm{rc}\left(\{x \mid Ax \leqslant b\}\right) = \{x \mid Ax \leqslant 0\} \qquad \mathrm{rc}\left(\{x \mid Ax \leqslant b\}\right)^\circ = \mathrm{Cone}(A^\top)\ . \tag{3}$$

Examples of these definition can be found in Fig. 2.

Proposition 5 (Basic properties of normal fan (see e.g. [LR08]))**.** *If P is a polyhedron, the normal fan $\mathcal{N}(P)$ is a* finite *collection of polyhedral cones (and in particular a* polyhedral complex*). Further, the support of $\mathcal{N}(P)$ can be expressed geometrically as the polar of the recession cone of P,* i.e.

$$\mathrm{supp}\,\mathcal{N}(P) = \big(\mathrm{rc}(P)\big)^\circ \tag{4}$$

2.3 Active constraints

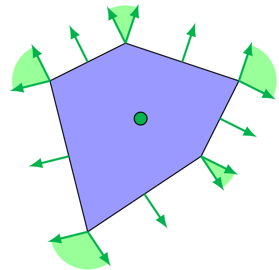
We introduce in this subsection the collection of active constraints which we use to obtain explicit formulas and make computations in practice. This notion is algebraic and depends on the matrix A and vector b used to defined a polyhedron, which we call *H-representation* (sometimes called external representation).

For any matrix $A \in \mathbb{R}^{q\times p}$ and a subset $I \subset [q]$, we denote by A_I the submatrix composed of the *rows* of indices in I of A

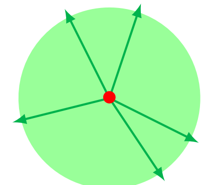
$$A_I := A_{I,\cdot} = (A_{i,j})_{i\in I, j\in[p]}$$

For $i \in [q]$, we also denote $A_i := A_{\{i\}}$ the ith row of A. To avoid confusion, we use the parenthesis rule $A_I^\top := (A_I)^\top$.

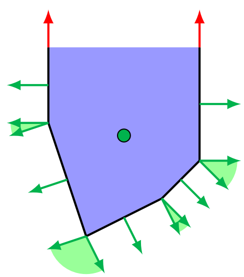
[2]Sometimes called *outer* normal cones and fan, as opposed to *inner* cones obtained either by inverting the inequality in the definition of the normal cone or by taking the opposite cones respect to the origin.



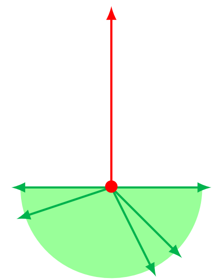
(a) A polytope P and its normal fan $\mathcal{N}(P)$



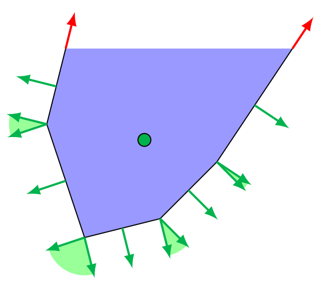
(b) The recession cone of $\operatorname{rc} P = \{0\}$ in red and its normal fan $\mathcal{N}(P)$ in green. $\operatorname{supp}(\mathcal{N}) = \mathbb{R}^2 = \{0\}^\circ$



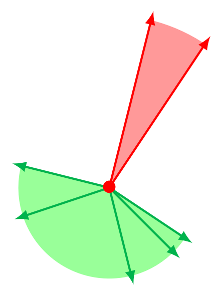
(c) The epigraph $E = epi(f)$ of a polyhedral function with a bounded domain and its normal fan $\mathcal{N}(E)$.



(d) The recession cone of $\operatorname{rc} E = \{0\} \times \mathbb{R}^+$ in red and its normal fan $\mathcal{N}(E)$ in green. $\operatorname{supp}(\mathcal{N}) = \mathbb{R} \times \mathbb{R}^- = (\{0\} \times \mathbb{R}^+)^\circ$



(e) A polyhedron P and its normal fan.



(f) The recession cone of P in red and its normal fan $\mathcal{N}(P)$ in green.

Figure 2: Examples of polyhedra and their normal fans and recession cones.

Definition 6 (Active constraints)**.** *For a polyhedron $P = \{x \mid Ax \leqslant b\}$, we denote by $I_{A,b}(x)$ the set of* active constraints *of P in $x \in \mathbb{R}^d$, with the H-representation $(A, b) \in \mathbb{R}^{q \times d} \times \mathbb{R}^q$:*

$$I_{A,b}(x) := \{i \in [q] \mid A_i x = b_i\}$$

More generally, for a set $E \subset P$, we set $I_{A,b}(E) := \bigcap_{x \in E} I_{A,b}(x)$.

We denote by $\mathcal{I}(A, b)$, the collection of sets of active constraints of P with the external representation (A, b):

$$\mathcal{I}(A, b) := \{I_{A,b}(x) \mid Ax \leqslant b\}$$

For a polyhedron P, we will denote by P^ψ the face of P given by $\arg\min_{x \in P} \psi^\top x$.

Proposition 7. *If P is a polyhedron, its normal fan $\mathcal{N}(P)$, its set of non-empty faces $\mathcal{F}(P) \setminus \{\emptyset\}$ and its collection of sets of active constraints $\mathcal{I}(P)$ are in one-to-one correspondence. Furthermore, the orders are preserved or inverted by the correspondences as indicated by figure 4.*

The proofs of such correspondences can be found in [LR08].

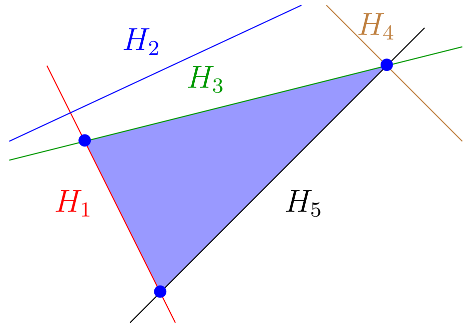


Figure 3: An example of a polyhedron $P = \{x \mid Ax \leqslant b\}$ with an H-representation (A, b). Each H_i corresponds to the hyperplane $\{x \mid A_i x = b_i\}$ and the label H_i is located in the halfspace $\{x \mid A_i x \geqslant b_i\}$. We have $\mathcal{I}(A, b) = \big\{\emptyset, \{1\}, \{1, 3\}, \{3\}, \{3, 4, 5\}, \{5\}, \{1, 5\}\big\}$. Constraint 2 is never active ($2 \notin \operatorname{supp} \mathcal{I}(A, b)$), and constraint 4 is redundant with constraints 3 and 5.

$$\begin{tikzcd}
& \boxed{F \in \mathcal{F}(P)\backslash\{\emptyset\}, \rhd} \arrow[dl, bend right, "I_{A,b}(F)"'] \arrow[dr, bend left, "N_P(F)"] & \\
\boxed{I \in \mathcal{I}(A,b), \subset} \arrow[ur, bend left, "\{x \in P \,\mid\, A_I x = b_I\}"] \arrow[rr, bend left, "\mathrm{Cone}(A_I^\top)"] & & \boxed{N \in \mathcal{N}(P), \lhd} \arrow[ul, bend right, "\bigcap_{\psi \in -N} P^\psi"'] \arrow[ll, bend left, "\{i \in \operatorname{supp} \mathcal{I}(A,b) \,\mid\, A_i^\top \in N\}"]
\end{tikzcd}$$

Figure 4: Monotonous one-to-one correspondences between normal fan, collection of active constraints sets and set of faces of a polyhedron $P = \{x \mid Ax \leqslant b\}$. For example, the downward arrow on the right reads $F_1 \lhd F_2$ is equivalent to $N_P(F_1) \rhd N_P(F_2)$.

2.4 Chamber complex

The affine regions of the cost-to-go function will correspond to cells of a chamber complex. The problems of projection of polyhedra, fibers and chambers complexes are studied in [BS92, RZ96, Ram96].

Definition 8 (Chamber complex)**.** *Let $P \subset \mathbb{R}^n$ be a polyhedron and π a linear projection of $\mathbb{R}^n$. For $x \in \pi(P)$ we define the* chamber of x for P along π *as*

$$\sigma_{P,\pi}(x) := \bigcap_{F \in \mathcal{F}(P) \text{ s.t. } x \in \pi(F)} \pi(F).$$

The chamber complex $\mathcal{C}(P, \pi)$ *of P along π is defined as the (finite) collection of chambers,* i.e.

$$\mathcal{C}(P, \pi) := \{\sigma_{P,\pi}(x) \mid x \in \pi(P)\}.$$

Further $\mathcal{C}(P, \pi)$ is a polyhedral complex such that $\operatorname{supp} \mathcal{C}(P, \pi) = \pi(P)$*. In particular, $\big\{ \operatorname{ri}(\sigma) \,|\, \sigma \in \mathcal{C}(P, \pi)\big\}$ is a partition of $\pi(P)$.*

More generally, the chamber complex of a polyhedral complex $\mathcal{P}$ is

$$\mathcal{C}(\mathcal{P}, \pi) := \{\sigma_{\mathcal{P},\pi}(x) \mid x \in \pi\big(\operatorname{supp}(\mathcal{P})\big)\}.$$

with $\sigma_{\mathcal{P},\pi}(x) := \bigcap_{F \in \mathcal{P} \text{ s.t. } x \in \pi(F)} \pi(F).$

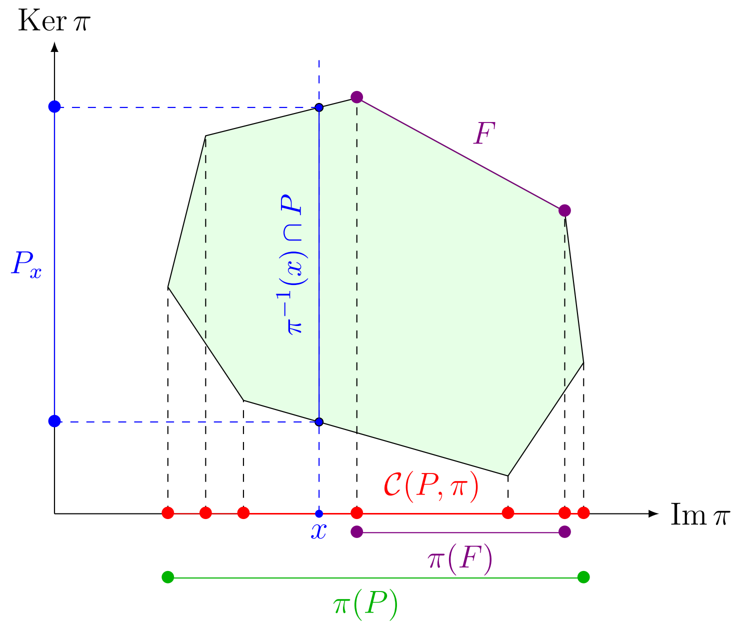


Figure 5: A polytope P in light green, its chamber complex in red on the x-axis and a fiber P_x in blue on the y-axis, for the orthogonal projection π on the horizontal axis.

Lemma 9 (Chamber complex monotonicity with respect to refinement order)**.** *Consider two polyhedral complexes of $\mathbb{R}^d$ and a projection π. If $\mathcal{R} \preccurlyeq \mathcal{S}$ then $\mathcal{C}(\mathcal{R},\pi) \preccurlyeq \mathcal{C}(\mathcal{S},\pi)$.*

Proof. For any $R \in \mathcal{R}$, there exist $S_R \in \mathcal{S}$ such that $R \subset S_R$. Let $x \in \operatorname{supp}\mathcal{C}(\mathcal{R},\pi) = \pi(\operatorname{supp}\mathcal{R}) = \pi(\operatorname{supp}\mathcal{S}) = \operatorname{supp}\mathcal{C}(\mathcal{S},\pi)$

$$\begin{aligned}\sigma_{\mathcal{R},\pi}(x) := \bigcap_{R\in\mathcal{R}\,\text{s.t. } x\in\pi(R)} \pi(R) &\subset \bigcap_{R\in\mathcal{R}\,\text{s.t. } x\in\pi(R)} \pi(S_R)\\ &\subset \bigcap_{S\in\mathcal{S}\text{s.t. } x\in\pi(S)} \pi(S) =: \sigma_{\mathcal{S},\pi}(x) \in \mathcal{C}(\mathcal{S},\pi)\end{aligned}$$

□

Recall that the *fiber* P_x of P along π at x is the projection of $P \cap \pi^{-1}(\{x\})$ on the space $\operatorname{Ker}(\pi)$ (see figure 5). An important property of a chamber complex is that all fibers are normally equivalent in each relative interior of cells of the chamber complex. More precisely, let $\sigma \in \mathcal{C}(P,\pi)$ be a chamber, and x and x' two points in its relative interior, then (see [BS92]), P_x and $P_{x'}$ are normally equivalent, i.e. they have the same normal fan $\mathcal{N}(P_x) = \mathcal{N}(P_{x'})$. Thus we define the normal fan $\mathcal{N}_\sigma$ above[3] $\sigma \in \mathcal{C}(P,\pi)$ by :

$$\mathcal{N}_\sigma := \mathcal{N}(P_x) \quad \text{for an arbitrary } x \in \operatorname{ri}(\sigma)$$

Part of the literature [CL98, LW97] uses the terms parameterized polyhedron for fiber and validity domain for chambers.

[3]The normal fan $\mathcal{N}_\sigma \subset 2^{\mathbb{R}^m}$ above σ should not be confused with $\mathcal{N}(\sigma) \subset 2^{\mathbb{R}^n}$ the normal fan of σ which will never appear in this paper.

3 Preserving polyhedrality with general cost distribution

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and $\boldsymbol{c} \in L_1(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^m)$ be an integrable random vector, and $\xi = (T, W, h)$ be deterministic. We study the cost-to-go function of Problem (2), which we recall here

$$V(x) := \mathbb{E}\left[\hat{V}(x, \boldsymbol{c})\right] \quad \text{with} \quad \hat{V}(x,c) := \begin{aligned}[t] \min_{y \in \mathbb{R}^m} \quad & c^\top y \\ \text{s.t.} \quad & Tx + Wy \leqslant h \end{aligned} \tag{5}$$

We denote the *coupling constraint polyhedron* of Problem (5) by

$$P := \{(x,y) \in \mathbb{R}^{n+m} \mid Tx + Wy \leqslant h\} \tag{6}$$

and π the projection of $\mathbb{R}^n \times \mathbb{R}^m$ onto $\mathbb{R}^n$ such that $\pi(x,y) = x$. The projection of P is the following polyhedron :

$$\pi(P) = \{x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^m,\ Tx + Wy \leqslant h\} \tag{7}$$

and for any $x \in \mathbb{R}^n$, the *fiber* of P along π is

$$P_x := \{y \in \mathbb{R}^m \mid Tx + Wy \leqslant h\} \tag{8}$$

3.1 Conditions for a well defined cost-to-go function

We want to find hypotheses under which V is defined without ambiguity and does not take value $-\infty$.

Proposition 10. *For every $x \in \mathbb{R}^n$ and $c \in \mathbb{R}^m$,*

$$\hat{V}(x,c) < +\infty \iff x \in \pi(P) \tag{9a}$$

$$\hat{V}(x,c) > -\infty \iff x \notin \pi(P) \ \textit{or} \ -c \in \operatorname{rc}(P_x)^\circ \tag{9b}$$

Furthermore, for all $x \in \pi(P)$,

$$\operatorname{rc}(P_x)^\circ = \operatorname{supp} \mathcal{N}(P_x) = \operatorname{Cone}(W^\top) \ .$$

Moreover the expectation in the definition of V is defined and takes value in $\mathbb{R} \cup \{+\infty\}$ if and only if $\operatorname{supp} \boldsymbol{c} \subset -\operatorname{Cone}(W^\top)$ *or* $\pi(P) = \emptyset$.

Proof. Eq. (9a) comes from the definitions of $\pi(P)$ in (7) and $\hat{V}(x,c)$ in (5). We now show Eq. (9b).

($\Rightarrow$) Let $x \in \pi(P)$ and $-c \notin \operatorname{rc}(P_x)^\circ$. By definition of the polar cone, there exists $r \in \operatorname{rc}(P_x)$ such that $-c^\top r > 0$. By definition of the recession cone, there exists $y_0 \in P_x$, such that, for any $\mu \in \mathbb{R}^+$, $y_0 + \mu r \in P_x$. Thus, we have $\lim_{\mu \to +\infty} c^\top (y_0 + \mu r) = -\infty$, and $\hat{V}(x,c) = \inf_{y \in P_x} c^\top y = -\infty$.

($\Leftarrow$) If $x \notin \pi(P)$, by Eq. (9a), $-\infty < \hat{V}(x,c)$. Let $x \in \pi(P)$ and $-c \in \operatorname{rc}(P_x)^\circ$. Then for all $r \in \operatorname{rc}(P_x)$, $-c^\top r \leqslant 0$, and $\min_{r \in \operatorname{rc}(P_x)} c^\top r = 0$. By Minkowski Weyl theorem (see e.g. [Zie12, 1.2]), there exists a polytope Q such that $P_x = Q + \operatorname{rc}(P_x)$. Thus, $\hat{V}(x,c) = \min_{y_0 \in Q, r \in \operatorname{rc}(P_x)} c^\top (y_0 + r) = \min_{y_0 \in Q} c^\top y_0$ is finite as Q is bounded.

We have that $\operatorname{rc}(P_x)^\circ = \operatorname{supp}(\mathcal{N}(P_x))$ by Eq. (4). Further, by Eq. (3) all non empty fibers have the same recession cone $\{y \mid Wy \leqslant 0\}$ whose polar is $\operatorname{Cone}(W^\top)$.

Note that $\hat{V}(x, \cdot)$ is Borel-measurable as the value function of a linear program (see [SDR14, 2.1.3]). Moreover, let $M = \max_{y_0 \in Q} \|y_0\|_\infty < \infty$, then for $c \in -\operatorname{Cone}(W^\top)$, $\hat{V}(x,c) \geqslant -M\|c\|_1$.

Thus, if $\operatorname{supp} \boldsymbol{c} \subset -\operatorname{Cone}(W^\top)$, then $\mathbb{E}\big[(-\hat{V}(x,\boldsymbol{c}))_+\big] < +\infty$ and the expectation $\mathbb{E}\big[\hat{V}(x,\boldsymbol{c})\big]$ is well defined with value in $\mathbb{R}\cup\{+\infty\}$.

Conversely if $\pi(P) \neq \emptyset$ and $\operatorname{supp} \boldsymbol{c} \not\subset -\operatorname{Cone}(W^\top)$, let $x \in \pi(P)$, then there exists a set $E \subset \mathbb{R}^m \backslash \operatorname{rc}(P_x)^\circ$ such that $\mathbb{P}\big[-\boldsymbol{c} \in E\big] > 0$. By Eq. (9b), $\hat{V}(x,c) = -\infty$ for all $c \in E$ so $\mathbb{E}\big[\hat{V}(x,\boldsymbol{c})\big]$ is either undefined or takes the value $-\infty$. □

With this property in mind, we make the following assumption:

Assumption 1. *The cost* $\boldsymbol{c} \in L^1(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^m)$ *is integrable with* $\operatorname{supp} \boldsymbol{c} \subset -\operatorname{Cone}(W^\top)$*.*

3.2 Reduction to a finite number of scenarios

We now show that the expectation in $V(x)$ can be reduced to a finite sum.

Recall that for any $x, x' \in \operatorname{ri}(\sigma)$, P_x and $P_{x'}$ are normally equivalent i.e. they have the same normal fan that we denote $\mathcal{N}_\sigma$. By the correspondences of proposition 7, for any x and $x' \in \operatorname{ri}(\sigma)$, P_x and $P_{x'}$ also have the same collection of sets of active constraints $\mathcal{I}(W, h-Tx) = \mathcal{I}(W, h-Tx')$. Thus, we define the collection of sets of active constraints of a chamber σ by :

$$\mathcal{I}_\sigma := \mathcal{I}(W, h-Tx) \quad \text{for an arbitrary } x \in \operatorname{ri}(\sigma) \tag{10}$$

Theorem 11. *Under Assumption 1, V is well-defined on $\mathbb{R}^n$ with value in $\mathbb{R}\cup\{+\infty\}$. Moreover, if $\sigma \in \mathcal{C}(P,\pi)$ is a chamber and $x \in \operatorname{ri}(\sigma)$ we have,*

$$\begin{aligned} V(x) &= \sum_{F\in\mathcal{F}(P_x)\backslash\{\emptyset\}} \mathbb{E}\big[\mathbb{1}_{\boldsymbol{c}\in-\operatorname{ri} N_{P_x}(F)}\boldsymbol{c}^\top\big] y_F && \text{(11a)}\\ &= \sum_{N\in\mathcal{N}_\sigma} \mathbb{E}\big[\mathbb{1}_{\boldsymbol{c}\in-\operatorname{ri} N}\boldsymbol{c}^\top\big] y_N(x) && \text{(11b)}\\ &= \sum_{I\in\mathcal{I}_\sigma} \mathbb{E}\big[\mathbb{1}_{\boldsymbol{c}\in-\operatorname{ri}\operatorname{Cone}(W_I^\top)}\boldsymbol{c}^\top\big] y_I(x) && \text{(11c)} \end{aligned}$$

where, if F is a face, y_F denotes an arbitrary point in F; if N is a cone, $y_N(x)$ denotes an arbitrary point in $\cap_{c\in -N} P_x^c$; and, if I is an active constraints sets, $y_I(x)$ denotes an arbitrary point in P_x verifying $T_I x + W_I y = h_I$.

Proof. If $x \notin \pi(P)$, $V(x)$ is well defined and equals $\mathbb{E}\big[\hat{V}(x,\boldsymbol{c})\big] = \mathbb{E}\big[+\infty\big] = +\infty$.

Let $x \in \pi(P)$ and $c \in \operatorname{supp}\mathcal{N}(P_x)$. If $-c \in \operatorname{ri}(N_{P_x}(F))$ then $-c \in N_{P_x}(F)$ which is equivalent to $c^\top y_F \leqslant c^\top y$ for all $y_F \in F$ and $y \in P_x$, *i.e.* $F \subset \arg\min_{y\in P_x} c^\top y = P_x^c$, thus $\hat{V}(x,c) = c^\top y_F$ for any $y_F \in F$. By Proposition 5 the normal fan is a polyhedral complex and thanks to the partition property of Proposition 2, we know that c belongs to one and only one relative interior of a normal cone, leading to

$$\hat{V}(x,c) = \sum_{F\in\mathcal{F}(P_x)\backslash\{\emptyset\}} \mathbb{1}_{-c\in\operatorname{ri} N_{P_x}(F)} c^\top y_F$$

By Assumption 1, $\boldsymbol{c}$ is integrable and $\operatorname{supp} \boldsymbol{c} \subset -\operatorname{Cone}(W^\top) = -\mathcal{N}(P_x)$, thus V is well defined and (11a) holds.

The other formulas are deduced from the correspondences of Proposition 7 applied to a fiber noting that, thanks to the normal equivalence property, the collection of active constraints and the normal fan depend only on the chamber σ whose relative interior contains x.

□

Note that when $\boldsymbol{c}$ is absolutely continuous respect to the Lebesgue measure of $\mathbb{R}^m$, in Eq. (11), we can restrict the sum on vertices, maximal cones or maximal active constraints sets and take the whole sets instead of their relative interiors. Fig. 9 illustrates this theorem.

The next lemma shows that we can replace a general random cost by a cost with finite support.

Lemma 12 (Quantization of the cost distribution)**.** *Under Assumption 1, let $\mathcal{R}$ be a refinement of $\bigwedge_{\sigma\in\mathcal{C}(P,\pi)} -\mathcal{N}_\sigma$, then*

$$V(x) = \sum_{R\in\mathcal{R}} \check{p}_R \hat{V}(x, \check{c}_R) \quad \textit{with} \quad \hat{V}(x,\check{c}_R) := \min_{y\in\mathbb{R}^m} \quad \check{c}_R^\top y + \mathbb{I}_{Tx+Wy\leqslant h} \tag{12}$$

where $\check{p}_R := \mathbb{P}\big[\boldsymbol{c}\in \mathrm{ri}(R)\big]$ and $\check{c}_R := \mathbb{E}\big[\boldsymbol{c} \,|\, \boldsymbol{c}\in\mathrm{ri}(R)\big]$ if $\check{p}_R > 0$ and $\check{c}_R := 0$ if $\check{p}_R = 0$.

Note that $\bigwedge_{\sigma\in\mathcal{C}(P,\pi)} \mathcal{N}_\sigma$ equals the chamber complex $\mathcal{C}(\mathcal{N}(P), \pi_y^{x,y})$ of the normal fan $\mathcal{N}(P)$ of the coupling constraint polyhedron along the projection $\pi_y^{x,y} : (x,y) \mapsto y$. Moreover, it is also the normal fan of the fiber polyhedron $\Sigma\big(P, \pi(P)\big)$ defined in [BS92].

Proof. Let $\sigma \in \mathcal{C}(P,\pi)$ and $x\in\mathrm{ri}(\sigma)$. For $R\in\mathcal{R}$, consider the set $\{N\in -\mathcal{N}_\sigma | \,\mathrm{ri}(R)\subset\mathrm{ri}(N)\}$. As $\mathcal{R}$ is a refinement of $-\mathcal{N}_\sigma$, this set is non empty. As $-\mathcal{N}_\sigma$ is a polyhedral complex, this set contains exactly one element that we denote $N(R)$. By Eq. (11b),

$$\begin{aligned}
V(x) &= \sum_{N\in-\mathcal{N}_\sigma} \mathbb{E}\Big[\sum_{R\in\mathcal{R}|\,\mathrm{ri}(R)\subset\mathrm{ri}(N)} \mathbb{1}_{\boldsymbol{c}\in\mathrm{ri}\,R}\, \boldsymbol{c}^\top\Big] y_N(x) && \text{by the partition property}\\
&= \sum_{R\in\mathcal{R}} \mathbb{E}\big[\mathbb{1}_{\boldsymbol{c}\in\mathrm{ri}\,R}\boldsymbol{c}^\top\big] y_{N(R)}(x) && \text{by linearity}\\
&= \sum_{R\in\mathcal{R}} \check{p}_R \check{c}_R^\top y_{N(R)}(x)\\
&= \sum_{R\in\mathcal{R}} \check{p}_R \min_{y\in\mathbb{R}^m} \check{c}_R^\top y + \mathbb{I}_{Tx+Wy\leqslant h}
\end{aligned}$$

by definition of $y_{N(R)}(x)$ and as $\check{c}_R\in N(R)$, which leads to Eq. (12). $\square$

Note that $\mathcal{R} = \bigwedge_{\sigma\in\mathcal{C}^{\max}(P,\pi)} -\mathcal{N}_\sigma$ satisfies the condition of Lemma 12 since if τ is a face of σ in the chamber complex, $\mathcal{N}_\sigma$ refines $\mathcal{N}_\tau$ by [RZ96, Lemma 2.2].

3.3 Explicit formula with barycentric coordinates

We define a coefficient $\mu(I)$ which can be interpreted as barycentric (conical) coordinates of the expectation of $\boldsymbol{c}$ in $-\operatorname{Cone}(W_I^\top)$. This section shows that the knowledge of the set $\mathcal{I}_\sigma$ of active constraints at σ (see (10)) and the $\mu(I)$ for $I\in\mathcal{I}_\sigma$ is sufficient to compute V.

Lemma 13. *Let σ be a chamber of $\mathcal{C}(P,\pi)$. For each set of active constraints $I\in\mathcal{I}_\sigma$, there exists a vector of positive coefficient $\mu(I)\in\mathbb{R}_+^I$ which satisfies*

$$-W_I^\top \mu(I) = \mathbb{E}\big[\boldsymbol{c}\mathbb{1}_{\boldsymbol{c}\in-\mathrm{ri}\,\mathrm{Cone}(W_I^\top)}\big] \tag{14}$$

For $x\in\mathrm{ri}\,\sigma$, under Assumption 1, the formula (11c) *can be rewritten as:*

$$V(x) = \sum_{I\in\mathcal{I}_\sigma} \mu(I)^\top (T_I x - h_I) \tag{15}$$

Proof. Note that $\mathbb{E}\big[\boldsymbol{c}\mathbb{1}_{\boldsymbol{c}\in-\operatorname{ri}\operatorname{Cone}(W_I^\top)}\big] \in -\operatorname{Cone}(W_I^\top) = -W_I^\top \mathbb{R}_+^I$, thus there exists $\mu(I)$ which verifies (14). For $x \in \operatorname{ri}\sigma$, (11c) becomes :

$$V(x) = \sum_{I\in\mathcal{I}_\sigma} -\mu(I)^\top W_I y_I(x) = \sum_{I\in\mathcal{I}_\sigma} \mu(I)^\top (T_I x - h_I)$$

□

We finally get the following theorem which gives an explicit formula of the polyhedral function V.

Theorem 14 (Master formula)**.** *Assume that $\boldsymbol{\xi} = (T, W, h)$ is deterministic, and Assumption 1 holds, then V, defined by (5), is a polyhedral function. Further, for all distributions of $\boldsymbol{c}$, V is affine on each cell of $\mathcal{C}(P,\pi)$, the chamber complex of the coupling constraint polyhedron P along the projection π on x.*

Finally, we have, for all $x \in \mathbb{R}^n$,

$$V(x) = \mathbb{I}_{x\in\pi(P)} + \max_{\sigma\in\mathcal{C}^{\max}(P,\pi)} \alpha_\sigma^\top x + \beta_\sigma \tag{16}$$

where $\alpha_\sigma := \sum_{I\in\mathcal{I}_\sigma} T_I^\top \mu(I)$, $\beta_\sigma := -\sum_{I\in\mathcal{I}_\sigma} h_I^\top \mu(I)$ and $\mu(I)$ satisfies Eq. (14)*.*

Proof. By Eq. (15), for all $x \in \operatorname{ri}(\sigma)$, $V(x) = \alpha_\sigma^\top x + \beta_\sigma$. Further as V is lower semicontinuous and convex (e.g. see [SDR14, prop. 2.7]), $V(x) = \alpha_\sigma^\top x + \beta_\sigma$, for all $x \in \sigma$. Suppose first $\dim\big(\pi(P)\big) = m$, then for $\sigma \in \mathcal{C}^{\max}(P,\pi)$, $x \to \alpha_\sigma^\top x + \beta_\sigma$ is a supporting affine function of V which coincide with V on σ whose dimension is m. Since $\bigcup_{\sigma\in\mathcal{C}^{\max}(P,\pi)} \sigma = \operatorname{supp}(\mathcal{C}(P,\pi)) = \pi(P)$, V is piecewise affine on the polyhedron $\pi(P)$ and equals to $+\infty$ elsewhere. Together with convexity of V, this yields Eq. (16). When $\pi(P)$ is not full dimensional, we get the same result by restraining the ambient space to the affine hull $\operatorname{Aff}\big(\pi(P)\big)$. Finally, since $\mathcal{C}(P,\pi)$ does not depend on $\boldsymbol{c}$, for all distributions of $\boldsymbol{c}$ satisfying *Assumption* 1, V is affine on each cell of $\mathcal{C}(P,\pi)$. □

We note that the terms of the master formula depends on the distribution of the random cost $\boldsymbol{c}$ only through the valuations $\mathbb{E}\big[\boldsymbol{c}\mathbb{1}_{\boldsymbol{c}\in-\operatorname{ri}N}\big]$ attached to the different cones N of the normal fan $\mathcal{N}_\sigma$.

We apply this master formula on an analytical example in Section 6. Note that when $\boldsymbol{c}$ admits a density, we have $\alpha_\sigma = \sum_{I\in\mathcal{I}_\sigma^{\max}} T_I^\top \mu(I)$ and $\beta_\sigma = -\sum_{I\in\mathcal{I}_\sigma^{\max}} h_I^\top \mu(I)$.

4 Polyhedral structure of MSLP

In this section, we show that the polyhedrality result established before for an expected cost-go-function with general cost distribution and deterministic constraints carry over to the case of stochastic constraints with finite support and then to multistage programming.

We denote by $\pi_x^{x,y}$ for the projection from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^n$ defined by $\pi_x^{x,y}(x', y') = x'$. The projections $\pi_{x,y}^{x,y,z}$, $\pi_x^{x,y,z}$, $\pi_y^{y,z}$, $\pi_{x_{t-1}}^{x_{t-1},z}$ are defined accordingly. Note that in the notation $\pi_x^{x,y,z}$, x, y and z are part of the notation and not parameters.

4.1 Propagating chamber complexes through Dynamic Programming

Before adapting Lemma 12 to include a recourse cost function, we start by a useful technical remark.

Recall that, for a polyhedron P and vector ψ, we denote $P^{\psi} := \arg\min_{x\in P} \psi^\top x$. Let f be a polyhedral function on $\mathbb{R}^d$, with a slight abuse of notation we denote $\operatorname{epi}(f)^{\psi,1} = \arg\min_{(x,z)\in \operatorname{epi}(f)} \psi^\top x + z$. We denote $\mathcal{F}_{\text{low}}\big(\operatorname{epi}(f)\big) := \{\operatorname{epi}(f)^{\psi,1} \mid \psi \in \mathbb{R}^d\}$ the set of *lower faces* of $\operatorname{epi}(f)$. The collection of projections (on $\mathbb{R}^d$) of lower faces of $\operatorname{epi}(f)$ is the coarsest polyhedral complex such that f is affine on each of its cell (see [DLRS10, Chapter 2]). Moreover, we have

$$\pi_{\mathbb{R}^d}\big((\operatorname{epi}(f)^{\psi,1}\big) = \arg\min_{x\in\mathbb{R}^d} \psi^\top x + f(x) \tag{17}$$

Lemma 15. *Let R be a polyhedral function on $\mathbb{R}^m$ and $\mathcal{R} := \pi_y^{y,z}\Big(\mathcal{F}_{\text{low}}\big(\operatorname{epi}(R)\big)\Big)$ the coarsest polyhedral complex such that R is affine on each element of $\mathcal{R}$. Let $\xi = (T, W, h)$ be fixed and Assumption 1 holds. Define, for all $x \in \mathbb{R}^n$*

$$\begin{aligned} Q(x,y) &:= R(y) + \mathbb{I}_{Tx+Wy\leqslant h} \\ V(x) &:= \mathbb{E}\big[\min_{y\in\mathbb{R}^m} \boldsymbol{c}^\top y + Q(x,y)\big] \end{aligned}$$

Let $\mathcal{V} := \mathcal{C}(\mathcal{F}(P)\wedge(\mathbb{R}^n\times\mathcal{R}), \pi_x^{x,y}) \subset 2^{\mathbb{R}^n}$ with $P := \{(x,y) \mid Tx + Wy \leqslant h\}$.

Then, $\mathcal{V} \preccurlyeq \mathcal{C}(\operatorname{epi}(Q), \pi_x^{x,y,z})$ and V is a polyhedral function which is affine on each element of $\mathcal{V}$.

Proof. We have $\operatorname{epi}(Q) = \big(\mathbb{R}^n \times \operatorname{epi}(R)\big) \cap (P\times\mathbb{R}) \subset \mathbb{R}^{n+m+1}$. Since

$$V(x) = \mathbb{E}\big[\min_{y\in\mathbb{R}^m, z\in\mathbb{R}} \boldsymbol{c}^\top y + z + \mathbb{I}_{(x,y,z)\in\operatorname{epi}(Q)}\big],$$

by Theorem 14 applied to the problem with variables (y,z) and the coupling polyhedron $\operatorname{epi}(Q)$, V is a polyhedral function affine on each element of $\mathcal{C}(\operatorname{epi}(Q), \pi_x^{x,y,z})$. We now show that $\mathcal{V} \preccurlyeq \mathcal{C}(\operatorname{epi}(Q), \pi_x^{x,y,z})$. As $\operatorname{epi}(Q)$ is the epigraph of a polyhedral function, $\mathcal{Q} := \pi_{x,y}^{x,y,z}\big(\mathcal{F}_{\text{low}}(\operatorname{epi}(Q))\big) \subset 2^{\mathbb{R}^{n+m}}$ is a polyhedral complex.

Let $x_0 \in \pi_x^{x,y,z}(\operatorname{epi}(Q))$, using notation of Definition 8,

$$\begin{aligned} \sigma_{\operatorname{epi}(Q),\pi_x^{x,y,z}}(x_0) &:= \bigcap_{F\in\mathcal{F}(\operatorname{epi}(Q))\text{ s.t. } x_0\in\pi_x^{x,y,z}(F)} \pi_x^{x,y,z}(F) \\ &= \bigcap_{F\in\mathcal{F}_{\text{low}}(\operatorname{epi}(Q))\text{ s.t. } x_0\in\pi_x^{x,y,z}(F)} \pi_x^{x,y,z}(F) \\ &= \bigcap_{F'\in\mathcal{Q}\text{ s.t. } x_0\in\pi_x^{x,y}(F')} \pi_x^{x,y}(F') =: \sigma_{\mathcal{Q},\pi_x^{x,y}}(x_0) \end{aligned}$$

Indeed, as $\operatorname{epi}(Q)$ is an epigraph of a polyhedral function, if $F \in \mathcal{F}(\operatorname{epi}(Q))$ such that $x_0 \in \pi_x^{x,y,z}(F)$ then there exists $G \in \mathcal{F}_{\text{low}}(\operatorname{epi}(Q))$ such that $G \lhd F$ and $x_0 \in \pi_x^{x,y,z}(G)$, allowing us to go from the first to second equality. The third equality is obtained by setting $F' = \pi_{x,y}^{x,y,z}(F)$. Thus, $\mathcal{C}(\operatorname{epi}(Q), \pi_x^{x,y,z}) = \mathcal{C}(\mathcal{Q}, \pi_x^{x,y})$.

We now show that $\mathcal{F}(P) \wedge (\mathbb{R}^n \times \mathcal{R}) \preccurlyeq \mathcal{Q}$. Let $G \in \mathcal{F}(P)\wedge\big(\mathbb{R}^n\times\mathcal{R}\big)$. There exist $\sigma \in \mathcal{R}$ and $F \in \mathcal{F}(P)$ such that $G = F \cap (\mathbb{R}^n \times \sigma)$. By definition of $\mathcal{F}_{\text{low}}$, there exists $\psi \in \mathbb{R}^m$ such

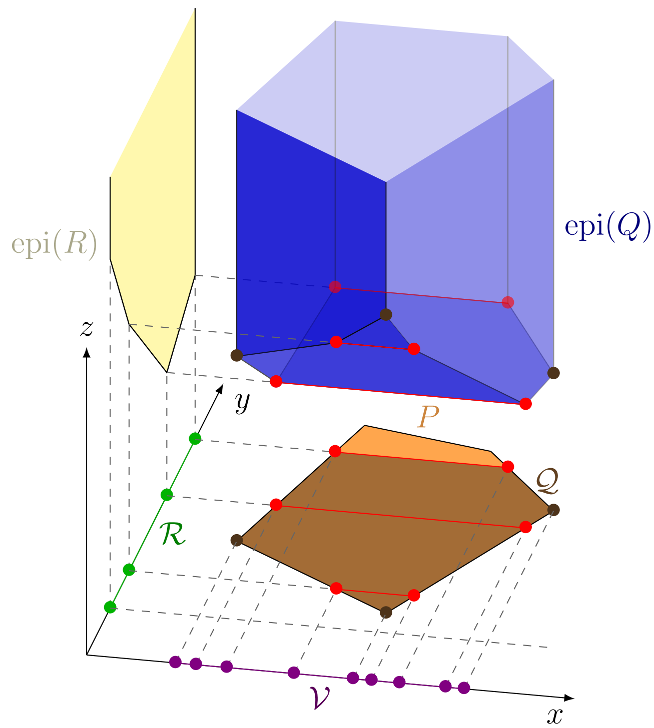


Figure 6: An illustration of the proof of Lemma 15 : the epigraph epi(Q) of the coupling function in blue in the (x, y, z) space, the epigraph of R in yellow in the (y, z) plane, the affine regions $\mathcal{R}$ of R in green on the y axis, the coupling polyhedron P in orange and brown in the (x, y) plane, the polyhedral complex $\mathcal{Q}$ in red and brown in the (x, y) plane and the chamber complex $\mathcal{V}$ in violet on the x axis.

that $\sigma = \pi_y^{y,z}\big(\operatorname{epi}(R)^{\psi,1}\big)$. We show that $G \subset \pi_{x,y}^{x,y,z}(\operatorname{epi}(Q)^{0,\psi,1}) \in \mathcal{Q}$. Indeed, let $(x, y) \in G = F \cap \big(\mathbb{R}^n \times \pi_y^{y,z}(\operatorname{epi}(R)^{\psi,1})\big)$. We have $(x, y) \in F \subset P$ such that $y \in \arg\min_{y' \in \mathbb{R}^m} \big\{\psi^\top y' + R(y')\big\}$. Which implies that $(x, y) \in \arg\min \big\{\psi^\top y' + R(y') \mid (x', y') \in P\big\}$. This also reads, by Eq. (17), as $(x, y) \in \pi_{x,y}^{x,y,z}(\operatorname{epi}(Q)^{0,\psi,1})$. Thus, $G \subset \pi_{x,y}^{x,y,z}(\operatorname{epi}(Q)^{0,\psi,1}) \in \mathcal{Q}$ leading to $\mathcal{F}(P) \wedge (\mathbb{R}^n \times \mathcal{R}) \preccurlyeq \mathcal{Q}$. Finally, by monotonicity, Lemma 9 ends the proof. □

Remark 16. *In Lemma 15, the complex $\mathcal{V}$ is independent of the distribution of $\boldsymbol{c}$. However, for special choices of $\boldsymbol{c}$, V might be affine on each cell of a coarser complex than $\mathcal{V}$. For instance, if $R = 0$ and $\boldsymbol{c} = 0$, we have that $V = \mathbb{I}_{\pi_x^{x,y}(P)}$, V is affine on $\pi_x^{x,y}(P)$. Nevertheless, $\mathcal{V} = \mathcal{C}(P, \pi_x^{x,y})$ is generally finer than $\mathcal{F}\big(\pi_x^{x,y}(P)\big)$.*

4.2 Exact quantization of MSLP

We next show that the multistage program with arbitrary cost distribution is equivalent to a multistage program with independent, finitely distributed, cost distributions. Further, for all step t, there exist affine regions, independent of the distributions of costs, where V_t is affine. Assumption 1 is naturally extended to the multistage setting as follows

Assumption 2. *We assume that the sequence $(\boldsymbol{c}_t, \boldsymbol{\xi}_t)_{t\in[t_{\max}]}$ is independent.*[4] *Further, for each*

[4]The results can be adapted to non-independent $\boldsymbol{\xi}_t$ as long as $\boldsymbol{c}_t$ is independent of $(\boldsymbol{c}_\tau)_{\tau<t}$ conditionally on $(\boldsymbol{\xi}_{\tau\leqslant t})$.

$t \in [t_{\max}]$, $\boldsymbol{\xi}_t = (\boldsymbol{T}_t, \boldsymbol{W}_t, \boldsymbol{h}_t)$ *is finitely supported, and* $\boldsymbol{c}_t \in L^1(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^{n_t})$ *is integrable with* $\mathbb{P}\big[\boldsymbol{c}_t \in -\operatorname{Cone}(\boldsymbol{W}_t^\top)\big] = 1$.

Note that Assumption 2 does not require independence between $\boldsymbol{c}_t$ and $\boldsymbol{\xi}_t$. For $t \in [t_{\max}]$, and $\xi = (T, W, h) \in \operatorname{supp}(\boldsymbol{\xi}_t)$ we define the coupling polyhedron

$$P_t(T, W, h) := \{(x_{t-1}, x_t) \in \mathbb{R}^{n_{t-1}} \times \mathbb{R}^{n_t} \mid Tx_{t-1} + Wx_t \leqslant h\},$$

and consider, for $x_{t-1} \in \mathbb{R}^{n_{t-1}}$,

$$\widetilde{V}_t(x_{t-1}|\xi) := \mathbb{E}\big[\min_{x_t \in \mathbb{R}^{n_t}} \boldsymbol{c}_t^\top x_t + V_{t+1}(x_t) + \mathbb{I}_{Tx_{t-1}+Wx_t \leqslant h} \mid \boldsymbol{\xi}_t = \xi\big]. \tag{19}$$

Then, the cost-to-go function V_t is obtained by

$$V_t(x_{t-1}) = \sum_{\xi \in \operatorname{supp}(\boldsymbol{\xi}_t)} \mathbb{P}\big[\boldsymbol{\xi}_t = \xi\big] \widetilde{V}_t(x_{t-1}|\xi) \tag{20}$$

The next two theorems extend the quantization results of Lemma 12 to the multistage settings.

Theorem 17 (Affine regions independent of the cost)**.** *Assume that* $(\boldsymbol{\xi}_t)_{t \in [t_{\max}]}$ *is a sequence of independent, finitely supported, random variables. We define by induction* $\mathcal{P}_{t_{\max}+1} := \{\mathbb{R}^{n_{t_{\max}}}\}$ *and for* $t \in [t_{\max}]$

$$\mathcal{P}_{t,\xi} := \mathcal{C}(\mathbb{R}^{n_t} \times \mathcal{P}_{t+1} \wedge \mathcal{F}\big(P_t(\xi)\big), \pi_{x_{t-1}}^{x_{t-1}, x_t})$$
$$\mathcal{P}_t := \bigwedge_{\xi_t \in \operatorname{supp} \boldsymbol{\xi}_t} \mathcal{P}_{t,\xi}$$

Then, for all costs distributions $(\boldsymbol{c}_t)_{t \in [t_{\max}]}$ *such that* $(\boldsymbol{c}_t, \boldsymbol{\xi}_t)_{t \in [t_{\max}]}$ *satisfies Assumption 2 and all* $t \in [t_{\max}]$, *we have* $\operatorname{supp}(\mathcal{P}_t) = \operatorname{dom}(V_t)$, *and* V_t *is polyhedral and affine on each cell of* $\mathcal{P}_t$.

Proof. We set for all $t \in [t_{\max} + 1]$, $\mathcal{V}_t := \pi_{x_{t-1}}^{x_{t-1}, z}\big(\mathcal{F}_{\text{low}}\big(\operatorname{epi}(V_t)\big)\big)$ the affine regions of V_t. As $V_{t_{\max}+1} \equiv 0$ is polyhedral and affine on $\mathbb{R}^{n_{t_{\max}}}$, we have $\mathcal{P}_{t_{\max}+1} = \mathcal{V}_{t_{\max}+1}$. Assume now that for $t \in [t_{\max}]$, V_{t+1} is polyhedral and $\mathcal{P}_{t+1}$ refines $\mathcal{V}_{t+1}$ (i.e. V_{t+1} is affine on each cell $\sigma \in \mathcal{P}_{t+1}$).

By Lemma 15, $\widetilde{V}_t(\cdot|\xi)$, defined in Eq. (19), is affine on each cell of $\mathcal{C}(\mathbb{R}^{n_t} \times \mathcal{V}_{t+1} \wedge \mathcal{F}\big(P_t(\xi)\big), \pi_{x_{t-1}}^{x_{t-1}, x_t})$ which is refined by $\mathcal{P}_{t,\xi} = \mathcal{C}(\mathbb{R}^{n_t} \times \mathcal{P}_{t+1} \wedge \mathcal{F}\big(P_t(\xi)\big), \pi_{x_{t-1}}^{x_{t-1}, x_t})$ by induction hypothesis and Lemma 9. Thus, by Eq. (20), V_t is affine on each cell of $\mathcal{P}_t$. In particular, V_t is polyhedral and $\mathcal{P}_t := \bigwedge_{\xi_t \in \operatorname{supp} \boldsymbol{\xi}_t} \mathcal{P}_{t,\xi}$ refines $\mathcal{V}_t$. Backward induction ends the proof. □

By Lemma 15, we have that $\mathcal{P}_{t,\xi} \preccurlyeq \mathcal{C}(\operatorname{epi}\big(Q_t^\xi\big), \pi_{x_{t-1}}^{x_{t-1}, x_t, z})$ where $Q_t^\xi(x_{t-1}, x_t) := V_{t+1}(x_t) + \mathbb{I}_{Tx_{t-1}+Wx_t \leqslant h_t}$. In particular, consider $\sigma \in \mathcal{P}_{t,\xi}$, then for all $x_{t-1} \in \operatorname{ri}(\sigma)$, all fibers $\operatorname{epi}(Q_t^\xi)_{x_{t-1}}$ are normally equivalent. We can then define $\mathcal{N}_{t,\xi,\sigma} := \mathcal{N}(\operatorname{epi}(Q_t^\xi)_{x_{t-1}})$ for an arbitrary $x_{t-1} \in \operatorname{ri}(\sigma)$.

The next result shows that we can replace the MSLP problem Eq. (1) by an equivalent problem with a discrete cost distribution. We elaborate further on this interpretation in Remark 19.

Theorem 18 (Quantization of the cost distribution, Multistage case)**.** *Assume that* $(\boldsymbol{\xi}_t)_{t \in [t_{\max}]}$ *is a sequence of independent, finitely supported, random variables. Then, for all costs distributions such that* $(\boldsymbol{c}_t, \boldsymbol{\xi}_t)_{t \in [t_{\max}]}$ *satisfies Assumption 2, for all* $t \in [t_{\max}]$, *all* $x_{t-1} \in \mathbb{R}^{n_{t-1}}$ *and all* $\xi \in \operatorname{supp}(\boldsymbol{\xi}_t)$, *we have a quantized version of Eq.* (19)*:*

$$\widetilde{V}_t(x_{t-1}|\xi) = \sum_{N \in \mathcal{N}_{t,\xi}} \check{p}_{t,N|\xi} \min_{x_t \in \mathbb{R}^{n_t}} \Big\{\check{c}_{t,N|\xi}^\top x_t + V_{t+1}(x_t) + \mathbb{I}_{Tx_{t-1}+Wx_t \leqslant h}\Big\}$$

where $\mathcal{N}_{t,\xi} := \bigwedge_{\sigma\in\mathcal{P}_{t,\xi}} -\mathcal{N}_{t,\xi,\sigma}$ *and for all* $\xi \in \operatorname{supp}(\boldsymbol{\xi}_t)$ *and* $N \in \mathcal{N}_{t,\xi}$ *we denote*

$$\check{p}_{t,N|\xi} := \mathbb{P}\big[\boldsymbol{c}_t \in \operatorname{ri} N \mid \ \boldsymbol{\xi}_t = \xi\big]$$

$$\check{c}_{t,N|\xi} := \begin{cases} \mathbb{E}\big[\boldsymbol{c}_t \mid \boldsymbol{c}_t \in \operatorname{ri} N, \boldsymbol{\xi}_t = \xi\big] & \textit{if } \mathbb{P}\big[\boldsymbol{\xi}_t = \xi, \boldsymbol{x} \in \operatorname{ri} N\big] \neq 0 \\ 0 & \textit{otherwise} \end{cases}$$

Proof. Since $\widetilde{V}_t(x_{t-1}|\xi) = \mathbb{E}\big[\min_{x_t\in\mathbb{R}^{n_t}, z\in\mathbb{R}} \boldsymbol{c}^\top x_t + z + \mathbb{I}_{(x_{t-1},x_t,z)\in\operatorname{epi}(Q_t^\xi)}\big]$ and $\mathcal{P}_{t,\xi}$ refines $\mathcal{C}(\operatorname{epi}\big(Q_t^\xi\big), \pi_{x_{t-1}}^{x_{t-1},x_t,z})$, by applying Lemma 12 with variables (x_t, z) and the coupling constraints polyhedron $\operatorname{epi}(Q_t^\xi)$, we deduce that the coefficients $(\check{p}_{t,N|\xi})_{N\in\mathcal{N}_{t,\xi}}$ and $(\check{c}_{t,N|\xi})_{N\in\mathcal{N}_{t,\xi}}$ satisfy

$$\widetilde{V}_t(x_{t-1}|\xi) = \sum_{N\in\mathcal{N}_{t,\xi}} \check{p}_{t,N|\xi} \min_{x_t\in\mathbb{R}^{n_t}, z\in\mathbb{R}} \Big\{\check{c}_{t,N|\xi}^\top x_t + z + \mathbb{I}_{(x_{t-1},x_t,z)\in\operatorname{epi}(Q_t^\xi)}\Big\}$$

as the deterministic coefficient before z is equal to its conditional expectation. □

Remark 19. *Theorem 18 can be seen as an exact discretization of Problem* (1) *where we replace the sequence of random cost* $(\boldsymbol{c}_t)_{t\in[t_{\max}]}$ *by a finitely supported sequence of discrete independent random variable* $(\check{\boldsymbol{c}}_t)_{t\in[t_{\max}]}$ *whose law is given by*

$$\mathbb{P}\big[\check{\boldsymbol{c}}_t = \check{c}_{N_t|\xi} \big| \ \boldsymbol{\xi}_t = \xi\big] = \check{p}_{N_t|\xi} \qquad \forall N_t \in \mathcal{P}_{t,\xi},\ \forall \xi \in \operatorname{supp}(\boldsymbol{\xi}_t),\ \forall t \in [t_{\max}]\ .$$

Indeed, define the cost-to-go functions $\check{V}_t$ *as in Eq.* (1) *where the random costs* $\boldsymbol{c}_t$ *are replaced by* $\check{\boldsymbol{c}}_t$*. In particular* $\check{V}_t(x_{t-1})$ *is the value of a MSLP over a finite scenario tree where a node at time* t *is given as a collection* $(N_\tau, \xi_\tau)_{\tau\leqslant t}$ *where* $N_\tau \in \mathcal{N}_{\tau,\xi_\tau}$ *and* $\xi_\tau \in \operatorname{supp}(\boldsymbol{\xi}_\tau)$*. Then we have, for all* $t \in [t_{\max}]$, $V_t = \check{V}_t$*. In particular, for any* $x_{t-1} \in \mathbb{R}^{n_{t-1}}$*, the optimal decision* x_t *taken at time* t *as solution of* $\hat{V}_t(x_{t-1}, c_t, \xi_t)$ *given in* (1)*, can also be obtained by solving*

$$\begin{aligned} \min_{x_t\in\mathbb{R}^{n_t}} \quad & c_t^\top x_t + \check{V}_{t+1}(x_t) \\ s.t. \quad & T_t x_{t-1} + W_t x_t \leqslant h_t \end{aligned}$$

5 Computing the valuations appearing in the master formula

In this section, we show that, for standard classes of distributions, we can evaluate the cost-to-go function at any point x. We even determine the epigraph of the cost-to-go function. We show that the explicit formulas of Theorem 11 can be expressed as valuations and centroids on the maximal cells of this complex, as $\mathbb{E}\big[\boldsymbol{c}\mathbb{1}_{\boldsymbol{c}\in E}\big] = \mathbb{E}\big[\boldsymbol{c} \,|\, \boldsymbol{c} \in E\big]\mathbb{P}\big[\boldsymbol{c} \in E\big]$ for all non-negligible subsets E of $\mathbb{R}^m$. The formulas for specific distributions are summed up in Table 1. These formulas are established in Sections 5.1–5.3. We point out that these formulas are only valid for *simplices* or *simplicial cones* S with $\dim(S) = \dim(\operatorname{supp} \boldsymbol{c})$.

So before computing $\mu(I)$, for $I \in \mathcal{I}_\sigma$, as a preliminary stage, we need to compute a triangulation of $-\operatorname{Cone}(W_I^\top) \cap \operatorname{supp}(\boldsymbol{c})$. Then, when the cost has a uniform distribution on a polytope, we can compute exactly volumes and centroids to obtain $\mu(I)$. When the cost has an exponential distribution on a cone, we use Brion's formula [Bri88] to obtain $\mu(I)$. We finally discuss the case of Gaussian distributions (easily adaptable to distributions with rotational symmetry). In this case we express $\mu(I)$ in terms of solid angles and spherical centroids.

Distribution	Uniform on polytope	Exponential	Gaussian
$d\mathbb{P}(c)$	$\frac{\mathbb{1}_{c\in Q}}{\mathrm{Vol}_d(Q)} d\mathcal{L}_{\mathrm{Aff}(Q)}(c)$	$\frac{e^{\theta^\top c}\mathbb{1}_{c\in K}}{\Phi_K(\theta)} d\mathcal{L}_{\mathrm{Aff}(K)}c$	$\frac{e^{-\frac{1}{2}c^\top M^{-2}c}}{(2\pi)^{\frac{m}{2}}\det M}dc$
Support	Polytope : Q	Cone : K	$\mathbb{R}^m$
$\mathbb{P}\big[\boldsymbol{c}\in S\big]$	$\dfrac{\mathrm{Vol}_d(S)}{\mathrm{Vol}_d(Q)}$	$\dfrac{\lvert\det(\mathrm{Ray}(S))\rvert}{\Phi_K(\theta)}\prod_{r\in\mathrm{Ray}(S)}\dfrac{1}{-r^\top\theta}$	$\mathrm{Ang}\left(M^{-1}S\right)$
$\mathbb{E}\big[\boldsymbol{c}\mid\boldsymbol{c}\in S\big]$	$\frac{1}{d}\sum_{v\in\mathrm{Vert}(S)}v$	$\left(\sum_{r\in\mathrm{Ray}(S)}\frac{-r_i}{r^\top\theta}\right)_{i\in[m]}$	$\frac{\sqrt{2}\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m}{2})}M\,\mathrm{Centr}\left(S\cap\mathbb{S}_{m-1}\right)$

Table 1: Probabilities and expectations arising from different cost distributions over simplicial cones or simplices $S\subset\mathrm{supp}(\boldsymbol{c})$ with $\dim S=\dim(\mathrm{supp}\,\boldsymbol{c})$, where $\mathcal{L}_A$ is the Lebesgue measure on an affine space A.

5.1 Uniform distributions on polytopes

(a) When $0\notin Q$

(b) When $0\in Q$

Figure 7: The normal inner fan $-\mathcal{N}_\sigma$ in green the support Q in orange, the sets $-W_i^\top\mathbb{R}^+\cap Q$ in red and the triangulations of each $-\mathrm{Cone}(W_I^\top)\cap Q$ in red, orange and black.

The *volume* of a polytope $Q\subset\mathbb{R}^m$ is the volume of P seen as a subset of the smallest affine space $\mathrm{Aff}(Q)$ it lives in. The volume of a full dimensional simplex S in $\mathbb{R}^d$ with vertices $v_1,\dots,v_{d+1}$ (see for example [GK94] 3.1) is

$$\mathrm{Vol}(S)=\frac{1}{n!}\lvert\det(v_1-v_{d+1},\cdots,v_d-v_{d+1})\rvert \tag{23}$$

The *centroid* of a non-empty polytope $Q\subset\mathbb{R}^m$ is

$$\mathrm{Centr}\left(Q\right):=\frac{1}{\mathrm{Vol}\,Q}\int_Q y\,d\mathcal{L}_{\mathrm{Aff}\,Q}(y) \tag{24}$$

For instance, the centroid of a simplex S of (non necessary full) dimension d is the equibarycenter of its vertices : $\mathrm{Centr}\left(S\right)=\frac{1}{d+1}\sum_{v\in\mathrm{Vert}(S)}v$.

Let Q be a polytope of dimension d. Assume that $\boldsymbol{c}$ is uniform on Q. Let $S\subset Q$ be a simplex with $\dim(S)=\dim(Q)$, then we have

$$\mathbb{P}\big[\boldsymbol{c}\in S\big]=\frac{\mathrm{Vol}_d\,S}{\mathrm{Vol}_d\,Q}\qquad\text{and}\qquad\mathbb{E}\big[\boldsymbol{c}|\boldsymbol{c}\in S\big]=\frac{1}{d+1}\sum_{v\in\mathrm{Vert}(S)}v$$

5.2 Exponential distributions on cones

Let P a (not necessarily bounded) polyhedron and $\theta \in \operatorname{ri}\big((\operatorname{rc} P)^\circ\big)$, we denote by $\Phi_P(\theta)$ the *exponential valuation* of P with parameter θ, *i.e.*$\Phi_P(\theta) := \int_P e^{\theta^\top c} d\mathcal{L}_{\mathrm{Aff}(P)}(c)$.

Proposition 20 (Brion's formula [Bri88]). *Let $S = \operatorname{Cone}\big(\operatorname{Ray}(S)\big)$ be a simplicial cone with $\operatorname{Ray}(S)$ a matrix whose columns are representative rays of S. Then for any $\theta \in \operatorname{ri} S^\circ$, the exponential valuation of S is*

$$\Phi_S(\theta) = |\det(\operatorname{Ray}(S))| \prod_{r \in \operatorname{Ray}(S)} \frac{1}{-\theta^\top r} \tag{25}$$

Let K be a (non necessarily simplicial) cone and $\theta \in \operatorname{ri} K^\circ$ a vector. Assume that $\boldsymbol{c}$ has the following exponential density :

$$d\mathbb{P}(c) := e^{\theta^\top c} \mathbb{1}_{c \in K} \frac{1}{\Phi_K(\theta)} d\mathcal{L}_{\mathrm{Aff}(K)}(c) \tag{26}$$

Let $S \subset K$ be a simplicial cone with $\dim S = \dim K$, by Brion's formula (25),

$$\mathbb{P}\big[\boldsymbol{c} \in S\big] = \frac{\Phi_S(\theta)}{\Phi_K(\theta)} = \frac{1}{\Phi_K(\theta)} |\det(\operatorname{Ray}(S))| \prod_{r \in \operatorname{Ray}(S)} \frac{1}{-r^\top \theta}$$

Further,

$$\mathbb{E}\left[\boldsymbol{c} \mathbb{1}_{\boldsymbol{c} \in S}\right] = \frac{1}{\Phi_K(\theta)} \int_S c e^{\theta^\top c} dc = \frac{\nabla \Phi_S(\theta)}{\Phi_K(\theta)}$$

Computing gradient coordinates we get

$$\frac{\partial \Phi_S(\theta)}{\partial \theta_i} = |\det(\operatorname{Ray}(S))| \sum_{r \in \operatorname{Ray}(S)} \Big(\prod_{r' \in \operatorname{Ray}(S) \setminus \{r\}} \frac{1}{-r'^\top \theta} \Big) \frac{r_i}{(r^\top \theta)^2} = \Phi_S(\theta) \sum_{r \in \operatorname{Ray}(S)} \frac{-r_i}{r^\top \theta}$$

Which finally yields

$$\mathbb{E}\big[\boldsymbol{c} \mid \boldsymbol{c} \in S\big] = \Big(\sum_{r \in \operatorname{Ray}(S)} \frac{-r_i}{r^\top \theta} \Big)_{i \in [m]}$$

5.3 Gaussian distributions

The *solid angle* of a pointed cone $K \subset \mathbb{R}^d$ is the volume of its intersection with the unit ball $\mathbb{B}_d$: $\operatorname{Ang}\big(K\big) := \frac{\operatorname{Vol}_d(K \cap \mathbb{B}_d)}{\operatorname{Vol}_d \mathbb{B}_d}$. Recall that $\operatorname{Vol}_d \mathbb{B}_d = \pi^{\frac{d}{2}} / \Gamma(\frac{d}{2} + 1)$ with Γ the Euler gamma function.

Proposition 21 ([Rib06]). *For any function $f : \mathbb{R}^m \to \mathbb{R}$ invariant under rotations around the origin and any pointed cone $K \subset \mathbb{R}^m$, we have $\operatorname{Ang}\big(K\big) \int_{\mathbb{R}^m} f = \int_K f$.*

Let $\boldsymbol{c}$ be a non-degenerated, centered, Gaussian random variable of variance M^2, where M is a symmetric positive definite matrix. Then, if K is a cone, by Proposition 21, we have

$$\mathbb{P}\big[\boldsymbol{c} \in K\big] = \int_{M^{-1}K} \frac{e^{-\frac{1}{2}\|c\|_2^2}}{(2\pi)^{\frac{m}{2}}} dc = \operatorname{Ang}\big(M^{-1}K\big)$$

We shall still use the notion of *centroid*, defined by Eq. (24), when K is not a polytope, but a subset of the sphere (then, the centroid generally does not belong to the sphere).

$$
\begin{aligned}
\mathbb{E}\left[\boldsymbol{c}\mathbb{1}_{\boldsymbol{c}\in K}\right] &= \int_{M^{-1}K} Mc\frac{e^{-\frac{1}{2}\|c\|_2^2}}{(2\pi)^{\frac{m}{2}}}dc = M\int_{\mathbb{R}^+} r^m\frac{e^{-\frac{r^2}{2}}}{(2\pi)^{\frac{m}{2}}}dr\int_{M^{-1}K\cap\mathbb{S}_{m-1}}\varphi d\varphi \\
&= M\frac{\Gamma(\frac{m+1}{2})}{\sqrt{2}\pi^{\frac{m}{2}}}\operatorname{Vol}_{m-1}(\mathbb{S}_{m-1})\operatorname{Ang}\left(M^{-1}K\right)\operatorname{Centr}\left(M^{-1}K\cap\mathbb{S}_{m-1}\right) \\
&= M\frac{\sqrt{2}\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m}{2})}\operatorname{Ang}\left(M^{-1}K\right)\operatorname{Centr}\left(M^{-1}K\cap\mathbb{S}_{m-1}\right)
\end{aligned}
$$

This reasoning can easily be adapted to distributions which are symmetric by rotation around the origin, up to a change of variable. For instance, we have formulas for uniform distributions on the volume and surface of ellipsoids as summed up in Table 2 when K is a full dimensional cone.

Description	$\mathbb{P}\left[\boldsymbol{c}\in K\right]$	$\mathbb{E}\left[\boldsymbol{c}\,\|\,\boldsymbol{c}\in K\right]$
Gaussian with variance M^2	$\operatorname{Ang}\left(M^{-1}K\right)$	$\frac{\sqrt{2}\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m}{2})}M\operatorname{Centr}\left(M^{-1}K\cap\mathbb{S}_{m-1}\right)$
Uniform on ellipsoid volume $M\mathbb{B}_m$	$\operatorname{Ang}\left(M^{-1}K\right)$	$\frac{m}{m+1}M\operatorname{Centr}\left(M^{-1}K\cap\mathbb{S}_{m-1}\right)$
Uniform on ellipsoid surface $M\mathbb{S}_{m-1}$	$\operatorname{Ang}\left(M^{-1}K\right)$	$M\operatorname{Centr}\left(M^{-1}K\cap\mathbb{S}_{m-1}\right)$

Table 2: Synthesis of formulas for some rotation symmetric distributions up to a change of variable

6 An analytical example

We present in this section an illustrative example. We consider the following cost-to-go function, with $n = 1$ and $m = 2$:

$$
V(x) = \mathbb{E}\begin{bmatrix}\min\limits_{y\in\mathbb{R}^2} & \boldsymbol{c}^\top y \\ \text{s.t.} & \|y\|_1 \leqslant 1, \quad y_1 \leqslant x \text{ and } y_2 \leqslant x\end{bmatrix}
$$

We show how to apply our results to compute an H-representation of $\operatorname{epi} V$.

The coupling polyhedron is $P = \{(x,y)\in\mathbb{R}^n\times\mathbb{R}^m \mid \|y\|_1 \leqslant 1,\, y_i \leqslant x \quad \forall i\in[m]\}$ presented in Fig. 8, and its V-representation is the collection of vertices $(0,-1,0)$, $(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$, $(0,0,-1)$, $(1,1,0)$, $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$, $(1,0,1)$ and the ray $(1,0,0)$. By projecting the different faces, we see that its projection is the half-line, $\pi(P) = [-\frac{1}{2},+\infty[$ and its chamber complex is $\mathcal{C}(P,\pi)$ is the collection of cells composed of $\{-\frac{1}{2}\}$, $[-\frac{1}{2},0]$, $\{0\}$, $[0,\frac{1}{2}]$, $\{\frac{1}{2}\}$, $[\frac{1}{2},1]$, $\{1\}$, $[1,+\infty)$ as presented in Fig. 8. As there are 4 different maximal chambers, there are 4 different classes of normally equivalent fibers as shown in Fig. 9. For each chamber, we can determine the collection of active constraints, for example $\mathcal{I}_{[-\frac{1}{2},0]} = \big\{\emptyset,\{5\},\{5,6\},\{6\},\{6,3\},\{3\},\{3,5\}\big\}$, with indices defined coherently with Fig. 10. We compute the $\mu(I)$ thanks to the formulas in Table 1. For example, when $\boldsymbol{c}$ is uniform on $\{y\in\mathbb{R}^2 \mid \|y\|_\infty \leqslant R\}$, Fig. 10 draws the regions whose areas and centroid need to be computed. We then deduce the affine forms $x \to \alpha_\sigma x + \beta_\sigma$ whose values are summed up in Table 3 and the graphs of V are drawn in Fig. 11 for different distribution of $\boldsymbol{c}$. We observe that the subdivision where V is affine is independent of the choice of the distribution of $\boldsymbol{c}$ in accordance with Theorem 14.

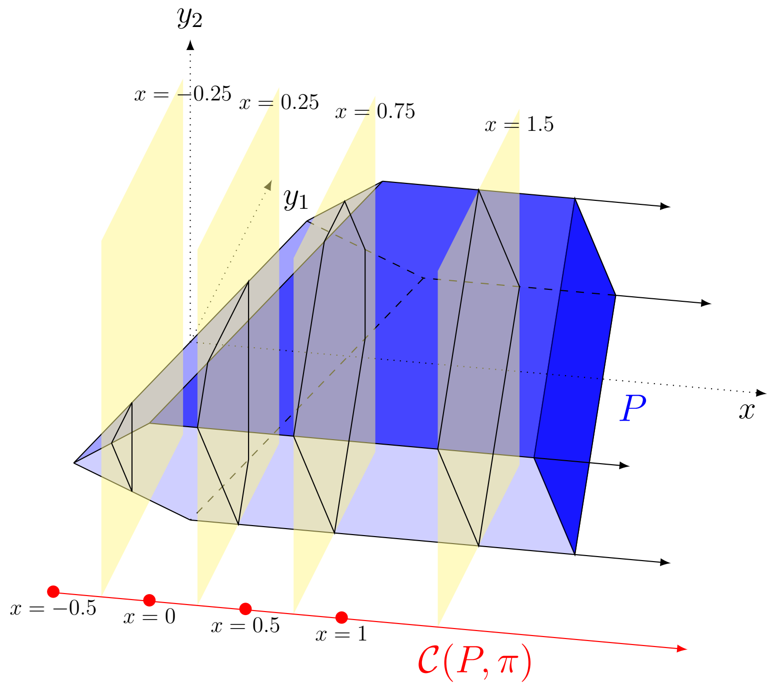
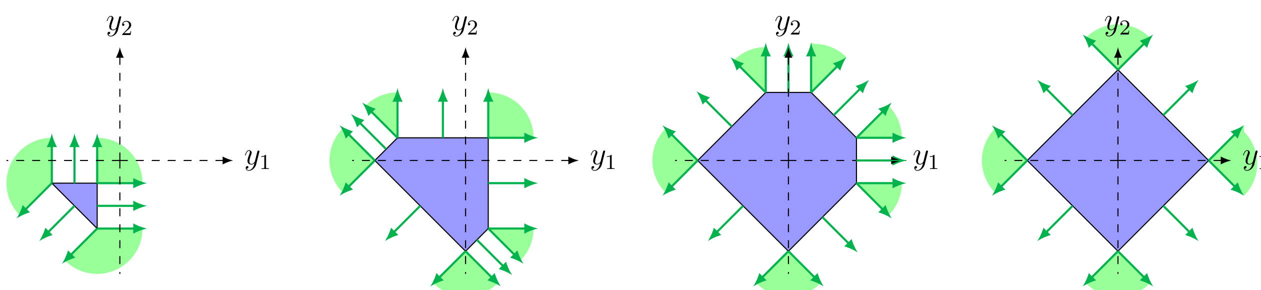


Figure 8: The coupling polyhedron P in blue, different cuts and fibers P_x vertical in yellow, and its chamber complex $\mathcal{C}(P, \pi)$ in red on the bottom.



(a) $x = -0.25, \sigma = [-0.5, 0]$ (b) $x = 0.25, \sigma = [0, 0.5]$ (c) $x = 0.75, \sigma = [0.5, 1]$ (d) $x \geqslant 1, \sigma = [1, +\infty)$

Figure 9: Fibers P_x in blue and their normal fan $\mathcal{N}(P_x) = \mathcal{N}_\sigma$ in green for different $x \in \mathbb{R}$

$d\mathbb{P}(c)$	$x < -\frac{1}{2}$	$-\frac{1}{2} \leqslant x \leqslant 0$	$0 \leqslant x \leqslant \frac{1}{2}$	$\frac{1}{2} \leqslant x \leqslant 1$	$1 \leqslant x$
$\frac{\mathbb{1}_{\\|c\\|_1 \leqslant R}}{2R^2} dc$	$+\infty$	$\frac{-R}{24}(7 + 14x)$	$\frac{-R}{24}(7 + 6x)$	$\frac{-R}{6}(2 + x)$	$\frac{-R}{2}$
$\frac{\theta^2 e^{-\theta \\|c\\|_1}}{4} dc$	$+\infty$	$\frac{-1}{8\theta}(7 + 14x)$	$\frac{-1}{8\theta}(7 + 6x)$	$\frac{-1}{2\theta}(2 + x)$	$\frac{-3}{2\theta}$
$\frac{\mathbb{1}_{\\|c\\|_\infty \leqslant R}}{4R^2} dc$	$+\infty$	$\frac{-R}{12}(5 + 10x)$	$\frac{-R}{12}(5 + 4x)$	$\frac{-R}{6}(3 + x)$	$\frac{-2R}{3}$
$\frac{e^{-\\|c\\|_2^2/2\gamma^2}}{2\pi\gamma^2} dc$	$+\infty$	$\frac{-\gamma(2+\sqrt{2})(1+2x)}{2\sqrt{2\pi}}$	$\frac{-\gamma(2+\sqrt{2}+2\sqrt{2}x)}{2\sqrt{2\pi}}$	$\frac{-2\gamma(1+(-1+\sqrt{2})x)}{\sqrt{2\pi}}$	$-\frac{2}{\sqrt{\pi}}\gamma$
$\frac{\mathbb{1}_{\\|c\\|_2 \leqslant R}}{\pi R^2} dc$	$+\infty$	$\frac{-R(2+\sqrt{2})(1+2x)}{3\pi}$	$\frac{-R(2+\sqrt{2}+2\sqrt{2}x)}{3\pi}$	$\frac{-4R(1+(-1+\sqrt{2})x)}{3\pi}$	$-\frac{4\sqrt{2}R}{3\pi}$

Table 3: Different values of $V(x)$ for different distribution of the cost $\boldsymbol{c}$

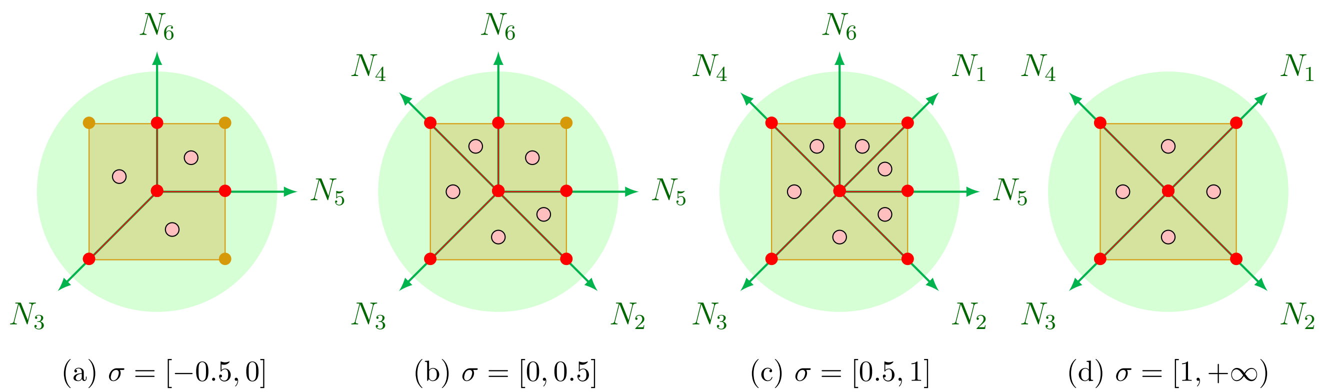


Figure 10: The normal fan $\mathcal{N}_\sigma$ in green with $N_i = W_i^\top \mathbb{R}^+$, $\boldsymbol{c}$ is uniform on the support $Q = -Q = B_\infty(0, R)$ in light orange, the sets $W_i^\top \mathbb{R}^+ \cap Q$ in red. The polyhedral complex $\mathcal{R}_\sigma$ is composed of the elements in red or orange. The barycenters $\mathbb{E}\big[\boldsymbol{c} | \boldsymbol{c} \in N\big]$ of the maximal cells are pink.

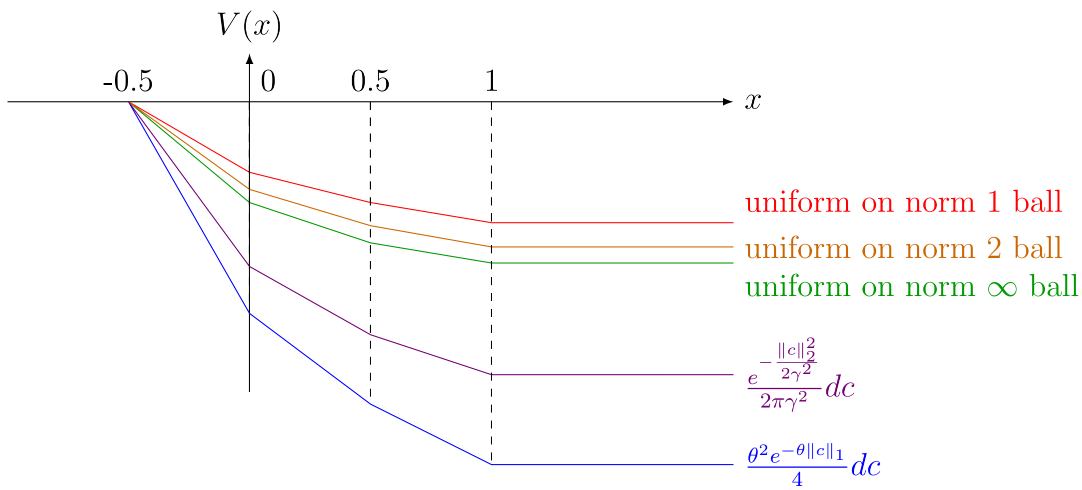


Figure 11: Graph of the cost-to-go function V for different distribution of the cost $\boldsymbol{c}$ with $R = \theta = \gamma = 1$.

7 Complexity

Hanasusanto, Kuhn and Wiesemann showed in [HKW16] that 2-stage stochastic programming is ♯P-hard, by reducing the computation of the volume of a polytope to the resolution of a 2-stage stochastic program.

We, on the other hand, show that for a fixed dimension of the recourse space, and a cost distribution uniform on a polytope, 2-stage programming is polynomial. Therefore, the status of 2-stage programming seems somehow comparable to the one of the computation of the volume of a polytope – which is also both ♯P-hard and polynomial when the dimension is fixed (see for example [GK94, 3.1.1]).

Further, when both the dimensions of the first and second stage decision spaces are fixed, we can not only evaluate, but compute an H-representation of V in polynomial time. These results also holds for exponential distributions on a cone. To show this, we rely on McMullen's and Stanley's upper bound theorems [McM70, Sta75]. The former theorem implies that number of vertices of a polyhedron can be polynomially bounded in terms of the number of facets, and vice-versa, *if the* *dimension is fixed*. The latter theorem implies that computing a triangulation takes a time that is polynomial in the number of vertices of the triangulation, still assuming that the dimension is fixed.

In order to study the complexity of MSLP we rely on the *Turing model* of computation (a.k.a. *bit model*) so we look for an exact solution, and all the computations are carried out with rational numbers.

7.1 Rationality and size of the cost-to-go functions

Recall that a polyhedron can be given in two manners. The "H-representation" provides an external description of the polyhedron, as the intersection of finitely many half-spaces. The "V-representation" provides an internal representation, writing the polyhedron as a Minkowski sum of a polytope (given as the convex hull of finitely many points) and of a polyhedral cone (generated by finitely many vectors).

We say that a polyhedron is *rational* if the inequalities in its H-representation are rational or, equivalently, the generators of its V-representation have rational coefficients. We shall say that a (convex) polyhedral function V is *rational* if its epigraph is a rational polyhedron.

Recall that, in the Turing model, the *size* (or encoding length see [GLS12, 1.3]) of an integer $k \in \mathbb{Z}$ is $\langle k \rangle := 1 + \lceil \log_2(|k| + 1) \rceil$; the size of a rational $r = \frac{p}{q} \in \mathbb{Q}$ with p and q coprime integers, is $\langle r \rangle := \langle p \rangle + \langle q \rangle$. The size of a rational matrix or a vector is the sum of the size of its entries. The size of an inequality $\alpha^\top x \leqslant \beta$ is $\langle \alpha \rangle + \langle \beta \rangle$. The size of a H-representation of a polyhedron is the sum of the sizes of its inequalities and the size of a V-representation of a polyhedron is the sum of the sizes of its generators.

If the dimension of the ambient space is *fixed*, one can pass from a H-representation to a V-representation in *polynomial time*, and vice versa. Indeed, the double description algorithm allows one to get a V-representation from a H-representation, see the discussion at the end of section 3.1 in [FP95], and use McMullen's upper bound theorem [McM70] and [GLS12, 6.2.4] to show that the computation time is polynomially bounded in the size of the H-representation. (A fortiori, the size of the V-representation is polynomially bounded in the size of the H-representation.) Dually, the same method allows one to obtain a H-representation from a V-representation. Hence, in the sequel, we shall use the term *size* of a polyhedron for the size of a V or H-representation: when dealing with polynomial-time complexity results in fixed dimension, whichever representation is used is irrelevant.

The following technical lemma yields a polynomial bound for the cardinalities of the collection of active constraints $\mathcal{I}_\sigma$ of a chamber σ (see Eq. (10)) and the triangulation of $-\operatorname{Cone}(W_I^\top) \cap \operatorname{supp}(\boldsymbol{c})$ arising in the representation of the cost-to-go function (15).

Lemma 22 (Polynomial cardinalities, for a fixed m)**.** *Assume that the recourse dimension m is fixed, that P has an H-representation given by a deterministic $\xi = (T, W, h)$ (as in* (6)*). Then, for an arbitrary $\sigma \in \mathcal{C}(P, \pi)$ and $x \in \operatorname{ri} \sigma$,*

1. *$\sharp\mathcal{I}_\sigma$ is polynomial in $\langle \xi \rangle$;*

2. *Let $Q \subset \mathbb{R}^m$ be a polytope, then for every $I \in \mathcal{I}_\sigma$, every triangulation of $-\operatorname{Cone}(W_I^\top) \cap Q$ has a polynomial cardinality in $\langle Q \rangle$ and $\langle \xi \rangle$;*

3. *Let $K \subset \mathbb{R}^m$ be a pointed cone, then for every $I \in \mathcal{I}_\sigma$, every triangulation of $-\operatorname{Cone}(W_I^\top) \cap K$ has a polynomial cardinality in $\langle K \rangle$ and $\langle \xi \rangle$;.*

Proof. The McMullen upper-bound theorem [McM70], in its dual version, guarantee that a polytope of dimension d with f facets has $O(n^{\lfloor d/2 \rfloor})$ faces. An easy proof of this asymptotic result is found in [Sei95].

Recall that $\xi = (T, W, h) \in \mathbb{R}^{q\times n} \times \mathbb{R}^{q\times m} \times \mathbb{R}^q$. Consider $\sigma \in \mathcal{C}(P, \pi)$ and $x \in \operatorname{ri}\sigma$. Observe that the fiber P_x (defined by Eq. (8)) is defined by q inequality and thus has at most q facets. Consequently, when m is fixed, there exists a constant κ_M such that $\sharp\mathcal{F}(P_x) \leqslant \kappa_M q^{\lfloor \frac{m}{2} \rfloor}$ where $q \leqslant \langle\xi\rangle$. Moreover, there is a one to one correspondence between the faces of P_x and the elements of $\mathcal{I}_\sigma$, which are sets of active constraints, see Proposition 7. This establishes the first assertion.

By the McMullen upper bound theorem, $-\operatorname{Cone}(W_I^\top)$ has at most $\kappa_M(\sharp I)^{\lfloor \frac{m}{2} \rfloor}$ facets. Thus, since $\sharp I \leqslant q \leqslant \langle\xi\rangle$, and as the intersection of two polyhedra is obtained as the aggregation of their H-representations, $-\operatorname{Cone}(W_I^\top) \cap Q$ has at most $\langle Q\rangle + \langle\xi\rangle$ facets and thus $\kappa_M(\langle Q\rangle + \langle\xi\rangle)^{\lfloor \frac{m}{2} \rfloor}$ vertices. By the Stanley upper bound theorem [Sta75], there exists κ_S such that the size of every triangulation with k vertices in dimension m is at most $\kappa_S k^{\lceil \frac{m+1}{2} \rceil}$, see [DLRS10, (2.6.3),(2.6.5)]. So every triangulation of $-\operatorname{Cone}(W_I^\top) \cap Q$ has a cardinality of at most $\kappa_S(\kappa_M(\langle Q\rangle + \langle\xi\rangle)^{\lfloor \frac{m}{2} \rfloor})^{\lceil \frac{m+1}{2} \rceil}$.

The proof of the third assertion is similar: replace vertices by rays. □

In the following lemmas we show that, in the uniform and exponential cost cases, the coefficients α_σ and β_σ are rational with an a priori bounded size. In particular, the polyhedral cost function V is rational.

Lemma 23 (Rationality and size of α_σ and β_σ in the exponential case)**.** *Let $\xi = (T, W, h)$ have rational coefficients and K be a rational cone. Assume that the distribution of the cost vector $\boldsymbol{c}$ is of exponential type, as in (26), with a rational parameter θ. Then, for all $\sigma \in \mathcal{C}(P, \pi)$ the coefficients α_σ and β_σ defined in Theorem 14 are rational. Further, we can compute in polynomial time in $\langle\xi\rangle$, $\langle K\rangle$ and $\langle\theta\rangle$ a bound φ, such that for all $\sigma \in \mathcal{C}(P, \pi)$, $\langle\alpha_\sigma\rangle \leqslant \varphi$ and $\langle\beta_\sigma\rangle \leqslant \varphi$.*

Proof. We introduced in Theorem 14 the notation $\alpha_\sigma := \sum_{I\in\mathcal{I}_\sigma} T_I^\top \mu(I)$ and $\beta_\sigma := -\sum_{I\in\mathcal{I}_\sigma} h_I^\top \mu(I)$. By Lemma 22, these sums have a number of terms that is polynomial in $\langle\xi\rangle$. It remains to show that, for $I \in \mathcal{I}_\sigma$, $\mu(I)$ can be chosen rational with a bounded size. Let $\mathcal{T}$ be a triangulation of $-\operatorname{Cone}(W_I^\top) \cap K$ constructed from it's V-representation. In particular, each $S \in \mathcal{T}^{\max}$ can be V-represented by rational rays. Thus, by the formula of Table 1 for each $S \in \mathcal{T}^{\max}$, $\mathbb{E}\big[\boldsymbol{c}\mathbb{1}_{\boldsymbol{c}\in \operatorname{ri} S}\big]$ is rational. Moreover, these formulas together with [GLS12, (1.3.3), (1.3.4)] give an algorithm to compute, in polynomial time, a bound of $\langle\mathbb{E}\big[\boldsymbol{c}\mathbb{1}_{\boldsymbol{c}\in \operatorname{ri} S}\big]\rangle$ which is polynomial in $\langle\xi\rangle$, $\langle\theta\rangle$ and $\langle K\rangle$. Since $\mathbb{E}\big[\boldsymbol{c}\mathbb{1}_{\boldsymbol{c}\in -\operatorname{ri}\operatorname{Cone}(W_I^\top)}\big] = \sum_{S\in\mathcal{T}} \mathbb{E}\big[\boldsymbol{c}\mathbb{1}_{\boldsymbol{c}\in \operatorname{ri} S}\big]$, by Lemma 22, we can also compute, in polynomial time, a polynomial bound of $\langle\mathbb{E}\big[\boldsymbol{c}\mathbb{1}_{\boldsymbol{c}\in -\operatorname{ri}\operatorname{Cone}(W_I^\top)}\big]\rangle$. Note that $\mu(I)$ can be chosen as a vertex of the rational polyhedron $\{\ \mu \in \mathbb{R}^I \mid \mu \geqslant 0,\ -W_I^\top \mu = \mathbb{E}\big[\boldsymbol{c}\mathbb{1}_{\boldsymbol{c}\in -\operatorname{ri}\operatorname{Cone}(W_I^\top)}\big]\}$. Then, by a standard result ([GLS12, 6.2.4]), such a vertex is rational and its size is polynomial in the size of the inequalities of the H-representation of the polyhedron.

□

Lemma 24 (Rationality and size of α_σ and β_σ in the uniform case)**.** *Let ξ have rational coefficients and Q be a rational polyhedron such that* $\operatorname{Aff}(Q) = \{y \in \mathbb{R}^m \mid \forall j \in J \subset [m], y_j = q_j \in \mathbb{Q}\}$ *where J is a subset of $[m]$ and q_j are rational numbers.(e.g. Q is full dimensional with $J = \emptyset$).*

Then, for all $\sigma \in \mathcal{C}(P, \pi)$ the coefficients α_σ and β_σ defined in Theorem 14 are rational. Further, we can compute in polynomial time in $\langle\xi\rangle$ and $\langle Q\rangle$ a bound φ, such that for all $\sigma \in \mathcal{C}(P, \pi)$, $\langle\alpha_\sigma\rangle \leqslant \varphi$ and $\langle\beta_\sigma\rangle \leqslant \varphi$.

Proof. Let $I \in \bigcup_{\sigma \in \mathcal{C}(P,\pi)} \mathcal{I}_\sigma$. Following the proof of Lemma 23, we need to show the rationality of $\mathbb{E}\big[\boldsymbol{c}\mathbb{1}_{\boldsymbol{c}\in -\operatorname{ri}\operatorname{Cone}(W_I^\top)}\big]$ and also to compute in polynomial time a bound for its size. Let $\mathcal{T}$ be a triangulation of $-\operatorname{Cone}(W_I^\top) \cap Q$. In particular, the vertices of $\mathcal{T}$ are vertices of $-\operatorname{Cone}(W_I^\top) \cap Q$, and so, they have rational coefficients. If $\dim(-\operatorname{Cone}(W_I^\top) \cap Q) < \dim Q$, then $\mathbb{E}\big[\boldsymbol{c}\mathbb{1}_{\boldsymbol{c}\in -\operatorname{ri}\operatorname{Cone}(W_I^\top)}\big] = 0$. Assume now that $\dim(-\operatorname{Cone}(W_I^\top) \cap Q) = \dim Q$.

Consider first, for simplicity, the special case in which Q thus $-\operatorname{Cone}(W_I^\top) \cap Q$ is full dimensional, so that every maximal simplex of $\mathcal{T}$ is full dimensional. By Eq. (23), the volume of a full dimensional simplex with rational vertices is rational. Then, the formulas of Table 1 yield the result. Assume now that $\operatorname{Aff}(Q) = \{y \in \mathbb{R}^m \mid \forall j \in J, y_j = q_j\}$ is not full dimensional, so that $\dim \operatorname{Aff}(Q) = k := m - \sharp J$. Every maximal simplices of $\mathcal{T}$ is full dimensional in $\operatorname{Aff}(Q)$. The same conclusion is obtained by considering the projection of these maximal simplices onto the affine space $\operatorname{Aff}(Q)$, and noting that this projection preserves the k-dimensional volume and the rational character of coordinates[5] , and does not increase the size. The formulas in Table 1 and Lemma 22 provides an algorithm to compute in polynomial time a bound for $\mathbb{E}\big[\boldsymbol{c}\mathbb{1}_{\boldsymbol{c}\in -\operatorname{ri}\operatorname{Cone}(W_I^\top)}\big]$.

□

For the Gaussian distribution, and the uniform distribution on an ellipsoid, the coefficients $\mu(I)$ can be determined in terms of solid angles (see [Rib06]) arising in Table 2. These coefficients are generally irrational.

Thus, to derive complexity results in the Turing model, we make the following assumption on the cost:

Assumption 3. *The random variable $\boldsymbol{\xi} = (\boldsymbol{T}, \boldsymbol{W}, \boldsymbol{h})$ is finitely supported with $p_\xi := \mathbb{P}\big[\boldsymbol{\xi} = \xi\big]$ for $\xi \in \operatorname{supp}(\boldsymbol{\xi})$. Further, for all $\xi \in \operatorname{supp}(\boldsymbol{\xi})$, $\boldsymbol{c}$ conditionally to $\{\boldsymbol{\xi} = \xi\}$ have a uniform distribution on a rational polytope $Q_\xi \subset -\operatorname{Cone}(W^\top)$ satisfying the assumption of Lemma 24 or a distribution exponential on a rational cone $K_\xi \subset -\operatorname{Cone}(W^\top)$ with rational parameter θ_ξ. We define $\langle \boldsymbol{c}_\xi \rangle$ as $\langle Q_\xi \rangle$ or $\langle K_\xi \rangle + \langle \theta_\xi \rangle$ accordingly.*

7.2 The 2-stage problem is polynomial-time with fixed dimension m

We start with a complexity result in the two-stage setting.

Lemma 25. *Under Assumption 3, given $\xi = (T, W, h)$ and Q_ξ or (K_ξ, θ_ξ) and consider $V(x) = \mathbb{E}\big[\min_{y\in\mathbb{R}^m} \boldsymbol{c} + \mathbb{I}_{Tx+Wy\leqslant h}\big]$. Then, when the dimension m is fixed, there exists an algorithm which returns, for an input $x \in \pi(P)$, $V(x)$ and $\alpha \in \partial V(x)$ in polynomial time in the size $\langle x \rangle + \langle \boldsymbol{c}_\xi \rangle + \langle \xi \rangle$.*

Proof. We first show that Algorithm 1 is polynomial. Line 1 takes a polynomial time by McMullen's theorem. The loop of Line 1 has a polynomial length by Lemma 22. Checking the conditions Line 1 is polynomial because computing a dimension of a polyhedron in H-representation is polynomial by [Fuk16, Thm 8.8]. Line 1 takes a polynomial time by [GK94, p390] and Lemma 22: the computation of the triangulation $\mathcal{T}$ with polynomial cardinality, takes a polynomial time. In Line 1 computing the formulas of Table 1 takes a polynomial time. Computing $\mu(I)$ in Line 1 reduces to solving a linear feasibility problem, which can be done in polynomial-time.

[5]This conclusion does not carry over to a general affine space. Indeed, the k-dimensional volume is obtained by applying Cayley Menger determinant formula (see for example [GK94, 3.6.1]). The latter formula outputs a number which is in a quadratic extension of the rationals numbers (i.e., generally, an irrational number). For example, if Δ_d is the canonical simplex $\{\lambda \in \mathbb{R}_+^{d+1} | \sum_{i=1}^{d+1} \lambda_i = 1\}$ then $\operatorname{Vol}(\Delta_d) = \frac{\sqrt{d+1}}{d!}$.

Data: An input $x \in \pi(P)$, $\xi = (T, W, h)$ and Q or (K, θ) describing the distribution of $\boldsymbol{c}$.
From the H-representation $(W, h - Tx)$ of P_x compute $\mathcal{I}(W, h - Tx)$ [FP95] ;
$\alpha \leftarrow 0 \in \mathbb{R}^n, \quad \beta \leftarrow 0 \in \mathbb{R}$;
for $I \in \mathcal{I}(W, h - Tx)$ **do**
  if $\dim\big(-\operatorname{Cone}(W_I^\top) \cap \operatorname{supp}(\boldsymbol{c})\big) = \dim \operatorname{supp}(\boldsymbol{c})$ **then**
    $E(I) \leftarrow 0$;
    Compute a triangulation $\mathcal{T}$ of $-\operatorname{Cone}(W_I^\top) \cap \operatorname{supp}(\boldsymbol{c})$;
    for $S \in \mathcal{T}^{\max}$ **do**
      $E(I) \leftarrow E(I) + \mathbb{P}\big[\boldsymbol{c} \in S\big]\mathbb{E}\big[\boldsymbol{c} \,|\, \boldsymbol{c} \in S\big]$ thanks to Table 1 ;
    end
    Compute $\mu(I)$ by solving the system $-W_I^\top \mu(I) = E(I)$ and $\mu(I) \geqslant 0$ (see Eq. (14));
    $\alpha \leftarrow \alpha + T_I^\top \mu(I), \quad \beta \leftarrow \beta - \mu(I)^\top h_I$;
  end
end
Return α and β;

Algorithm 1: Oracle returning α_σ and β_σ defined in Theorem 14 with $\sigma \in \mathcal{C}(P, \pi)$ and $x \in \operatorname{ri}(\sigma)$.

If there exists $\sigma \in \mathcal{C}^{\max}(P, \pi)$ such that $x \in \operatorname{ri}(\sigma)$ by Eq. (11), Algorithm 1 computes $V(x) = \alpha_\sigma^\top x + \beta_\sigma$ and $\alpha_\sigma \in \partial V(x)$. Indeed, if $\dim \pi(P) = \dim \sigma = m$, then by Eq. (15) for all $x \in \operatorname{ri} \sigma$, $V(x) = \alpha_\sigma^\top x + \beta_\sigma$, thus α_σ is a gradient of V. Otherwise, by restraining to the ambient space to $\operatorname{Aff}\big(\pi(P)\big)$, we see that α_σ is a subgradient of V. Therefore, for a generic $x \in \pi(P)$, Algorithm 1 allows us to compute the value and a subgradient of V in polynomial time. We next reduce to the generic case by a perturbation argument. More precisely, instead of applying Algorithm 1 to x, we perturb x to find a generic point x', close to x, such that there exists $\sigma \in \mathcal{C}^{\max}(P, \pi)$ with $x' \in \operatorname{ri}(\sigma)$ and $x \in \sigma$.

Since $x \in \pi(P)$, we can find $y \in \mathbb{R}^m$in polynomial time (by solving a linear system) such that $(x, y) \in P$. By [GLS12, 6.5.5], we can also find, still in polynomial time, a point $(\overline{x}, \overline{y}) \in \operatorname{ri}(P)$. We define, for any $\varepsilon \in \mathbb{R}$, $x(\varepsilon) := x + \varepsilon\big(\overline{x} - x + (\varepsilon, \varepsilon^2, \cdots, \varepsilon^n)\big)$ and $y(\varepsilon) := y + \varepsilon(\overline{y} - y)$.

Observe that there is $\varepsilon_0 > 0$ such that, for all $0 < \varepsilon < \varepsilon_0$, $\big(x(\varepsilon), y(\varepsilon)\big)$ does not cross any hyperplane of $\mathbb{R}^{n+m}$ defined by $T_i x + W_i y = h_i$ (supporting hyperplanes of P) and $x(\varepsilon)$ does not cross any hyperplane defining the chamber complex $\mathcal{C}(P, \pi)$. Indeed, the values of ϵ for which such a crossing occurs are roots of polynomials of degree at most $n + 1$. Moreover, since $(\overline{x}, \overline{y})$ is an interior point of P, we have $\big(x(\varepsilon), y(\varepsilon)\big) \in P$. It follows that $x(\epsilon)$ stays in a fixed cell σ of maximal dimension, for all $0 \leqslant \varepsilon < \varepsilon_0$.

Moreover, we can compute in polynomial time an explicit value of ε_0 using a classical bound, going back to Lagrange and Cauchy, for the smallest modulus of a non-zero root of a polynomial (see e.g. Th. 8.1.4 and Prop. 8.1.6 in [RS02]). We conclude by applying Algorithm 1 to $x' := x(\varepsilon_0/2)$. □

Theorem 26 ((2SLP) is polynomial-time for fixed m)**.** *Consider the problem*

$$\min_{x \in \mathbb{R}^n} \quad c_0^\top x + \mathbb{I}_{Ax \leqslant b} + \mathbb{E}\big[\min_{y \in \mathbb{R}^m} \boldsymbol{c}^\top y + \mathbb{I}_{\boldsymbol{T}x + \boldsymbol{W}y \leqslant \boldsymbol{h}}\big] \tag{2SLP}$$

Suppose that the recourse dimension m is fixed, and Assumption 3 holds.

Then, there exists an algorithm that solves (2SLP) in polynomial time in the input size $\langle c_0 \rangle + \langle A \rangle + \langle b \rangle + \sum_{\xi \in \operatorname{supp}(\boldsymbol{\xi})} \langle \boldsymbol{c}_\xi \rangle + \langle \xi \rangle + \langle p_\xi \rangle$.

Proof. We start with the case where $\boldsymbol{\xi}$ is deterministic. Recall that a *separation oracle* for a convex set $C \subset \mathbb{R}^d$ takes as argument a vector $x \in \mathbb{R}^d$, and either states that $x \in C$ or returns an hyperplane (strictly) separating x from C. We show that Algorithm 1 provides a polynomial-time separation oracle for $E := \mathrm{epi}(V) \cap (\{x \mid Ax \leqslant b\} \times \mathbb{R})$.

Let $(x,z) \in \mathbb{R}^{n+1}$. Suppose that there exists $i \in [q]$ such that $A_i x > b_i$ then E is separated from (x,z) by $\{(x',z') \mid A_i x' = \frac{b_i + A_i x}{2}\}$. Otherwise we have $Ax \leqslant b$, and by solving the dual of $\min_{y\in\mathbb{R}^q}\{\, 0 \mid Wy \leqslant h - Tx\}$, in polynomial time, we either find an unbounded ray generated by $\lambda \in \mathbb{R}^q$ such that $\lambda \geqslant 0$, $\lambda^\top W = 0$ and $\lambda^\top(h - Tx) < 0$ or a $y \in \mathbb{R}^m$ such that $Wy \leqslant h - Tx$, *i.e.* $x \in \pi(P)$, In the first case we have a *feasibility cut* $\{(x',z') \in \mathbb{R}^{n+1} \mid \lambda^\top T x' = \frac{\lambda^\top h + \lambda^\top T x}{2}\}$ separating E from (x,z). In the second case we have that $x \in \pi(P)$. By Lemma 25, we can compute $V(x)$ and a subgradient $\alpha \in \partial V(x)$ in polynomial time. If $z \geqslant V(x)$, $(x,z) \in E$ otherwise $\{(x',z') \mid \alpha^\top(x'-x) + \frac{V(x)+z}{2} = z'\}$ separates (x,z) from E.

We use this separation oracle to optimize the linear program $\min_{(x,z)\in E} c_0^\top x + z$, which is equivalent to (2SLP), in polynomial time through [GLS12, Theorem 6.4.9]. This theorem applies to the class of "well described" polyhedra (see [GLS12, 6.2.2]), which are rational polyhedra equipped with the dimension of the ambient space and an apriori bound on the encoding length of each facet-defining inequality. In order to apply it here, it remains to show that we can bound apriori the size of each inequalities in an H-representation of E.

Note that $E = \{(x,z) \in \mathbb{R}^{n+1} \mid Ax \leqslant b,\ x \in \pi(P),\ \alpha_\sigma x + \beta_\sigma \leqslant z,\ \forall \sigma \in \mathcal{C}^{\max}(P,\pi)\}$. By Lemma 22, there exists φ such that for all $\sigma \in \mathcal{C}^{\max}(P,\pi)$. $\langle \alpha_\sigma \rangle \leqslant \varphi$ and $\langle \beta_\sigma \rangle \leqslant \varphi$. Thus $(E, n+m+1, 2\varphi)$ is a well-described polyhedron, by [GLS12, 6.4.9] , (2SLP), under Assumption 3 is oracle-polynomial-time, and thus, polynomial-time.

To extend this result to allow for stochastic constraints, for each of the $\xi = (T,W,h) \in \mathrm{supp}(\boldsymbol{\xi})$ we can check if $x \in \pi\big(P(\xi)\big)$ and then use Lemma 25 to compute $\widetilde{V}(x|\xi) := \mathbb{E}\big[\hat{V}(x,\boldsymbol{c},\boldsymbol{\xi}) \mid \boldsymbol{\xi} = \xi\big]$ and $\alpha_\xi \in \partial\widetilde{V}(x|\xi)$, then $V(x) = \sum_{\xi\in\mathrm{supp}\,\boldsymbol{\xi}} p_\xi \widetilde{V}(x|\xi)$ and $\alpha = \sum_{\xi\in\mathrm{supp}\,\boldsymbol{\xi}} p_\xi \alpha^\xi \in \partial V(x)$. This yields a polynomial time separation oracle for $E := \mathrm{epi}(V) \cap (\{x \mid Ax \leqslant b\} \times \mathbb{R})$ and we conclude as previously. □

7.3 Determining V is polynomial-time with fixed dimensions n and m

In this section, we focus on the problem of computing a H-representation of the epigraph of the cost-to-go function. We next show that this can still be done in polynomial time, but we now need to fix *both* the dimensions n and m. This is a key step needed to extend our results to the multistage case (Section 7.4 below).

Theorem 27 (Computing V is polynomial for fixed n, m and $\sharp\,\mathrm{supp}(\boldsymbol{\xi})$)**.** *We consider the cost-to-go function*

$$V(x) := \mathbb{E}\big[\min_{y\in\mathbb{R}^m} \boldsymbol{c}^\top y + \mathbb{I}_{\boldsymbol{T}x+\boldsymbol{W}y\leqslant\boldsymbol{h}} + R(y)\big]$$

where R is a rational polyhedral function. Assume that n, m and $\sharp\,\mathrm{supp}(\boldsymbol{\xi})$ are fixed integers and $(\boldsymbol{c},\boldsymbol{\xi})$ satisfies Assumption 3.

Then, there exists an algorithm that find an H-representation of $\mathrm{epi}(V)$ in polynomial time in the input size $\langle \mathrm{epi}\, R\rangle + \sum_{\xi\in\mathrm{supp}(\boldsymbol{\xi})} \langle \boldsymbol{c}_\xi \rangle + \langle \xi \rangle + \langle p_\xi \rangle$.

Proof. We first focus on the case where $\boldsymbol{\xi} = \xi = (T,W,h)$ is deterministic and $R \equiv 0$. The "Master formula" (15) provides a H-representation of V consisting of "vertical" halfspaces given by facets of $\pi(P)$ and halfspaces $\{(x,z) \in \mathbb{R}^n \times \mathbb{R} \mid \alpha_\sigma^\top x + \beta_\sigma \leqslant z\}$. We next show that this H-representation can be computed in polynomial time.

The number of chambers in $\mathcal{C}(P,\pi)$ is polynomial when both n and m are fixed by [VWBC05, 3.9]. Thus computing the (maximal) chambers of the complex is polynomial in fixed dimension thanks to the algorithm in [CL98, 3.2]. For each cell $\sigma \in \mathcal{C}^{\max}(P,\pi)$, we can compute[6] in polynomial time α_σ and β_σ, by adapting Algorithm 1 (for example by taking $x \in \operatorname{ri} \sigma$). Further, the generators of a V-representation of $\pi(P)$ can be obtained by projecting the generators of a V-representation of P. Then, the double description method yields, in polynomial-time, an H-representation of $\pi(P)$.

Assume now that R is not necessarily equal to 0 and let $E := \{(x,y,z) \mid Tx + Wy \leqslant h, R(y) \leqslant z\}$. Since $\mathbb{E}\big[\min_{y \in \mathbb{R}^m} \boldsymbol{c}^\top y + \mathbb{I}_{Tx+Wy\leqslant h} + R(y)\big] = \mathbb{E}\big[\min_{y\in\mathbb{R}^m, z \in \mathbb{R}} \boldsymbol{c}^\top y + z + \mathbb{I}_{(x,y,z)\in E}\big]$ we can apply the previous result with the coupling constraint polyhedron E and the random variable $(\boldsymbol{c}, 1)$.

In order to extend this result to stochastic constraints, we compute, for each $\xi \in \operatorname{supp} \boldsymbol{\xi}$, the epigraph of $\operatorname{epi} \widetilde{V}(\cdot \mid \xi)$ where $\widetilde{V}(x|\xi) := \mathbb{E}\big[\hat{V}(x, \boldsymbol{c}, \boldsymbol{\xi}) \mid \boldsymbol{\xi} = \xi\big]$. We then compute V by intersecting and projecting polyhedra of dimension $n + m + \sharp \operatorname{supp}(\boldsymbol{\xi}) + 1$ as $\operatorname{epi} V = \{(x,z) \mid \exists (z_\xi)_{\xi \in \operatorname{supp} \boldsymbol{\xi}} \in \mathbb{R}^s, \sum_{\xi \in \operatorname{supp} \boldsymbol{\xi}} z_\xi = z, (x, z_\xi) \in \operatorname{epi}\big(\widetilde{V}(\cdot|\xi)\big)\}$. Since, n, m and $\sharp \operatorname{supp}(\boldsymbol{\xi})$ are fixed this computation is carried out in polynomial time.

□

7.4 Multistage programming with fixed horizon and dimensions is polynomial-time

Theorem 28 (MSLP is polynomial for fixed dimensions)**.** *Consider the MSLP problem with value $\hat{V}_1(x_0, c_1, \xi_1)$ as defined by (1). Assume that $t_{\max} \geqslant 3, n_2, \dots, n_{t_{\max}}$, $\sharp(\operatorname{supp} \boldsymbol{\xi}_2)$, $\cdots$,$\sharp(\operatorname{supp} \boldsymbol{\xi}_{t_{\max}})$ are fixed integers and for all $t \in [t_{\max}]$, $(\boldsymbol{c}_t, \boldsymbol{\xi}_t)$ satisfies Assumption 3. Then, there exists an algorithm that solves MSLP in polynomial time in the input size $\langle x_0 \rangle + \langle c_1 \rangle + \langle \xi_1 \rangle + \langle h_1 \rangle + \sum_{t=2}^{t_{\max}} \sum_{\xi \in \operatorname{supp}(\boldsymbol{\xi}_t)} \langle \boldsymbol{c}_{t,\xi} \rangle + \langle \xi \rangle + \langle p_{t,\xi} \rangle$.*

Proof. The inequality $z \geqslant 0$ provides a H-representation of $V_{t_{\max}+1}$. Assume that for $t \in \{3, \dots, t_{\max}\}$, V_{t+1} is a rational polyhedral function and we have computed a representation of V_{t+1} whose size is polynomially bounded in the size of the input. By Theorem 27, we compute V_t in polynomial time and its size is polynomial in the input. We apply recursively this procedure, the total time of computation and the size of V_2 are then bounded by a composition of $t_{\max} - 2$ polynomes. As in the proof of Theorem 26, we leverage the theory of linear programming with oracle and obtain that solving the MLSP problem takes a polynomial time in the size of the input when the horizon and dimensions are fixed. □

[6] This algorithm is not the most efficient one. Instead, we may enumerate the active constraints sets $I \in \bigcup_{\sigma \mathcal{C}^{\max}(P,\pi)} \operatorname{supp} \mathcal{I}_\sigma$ by looking at the n-faces of P (see algorithm [LW97, 4.2]), compute all the $\mu(I)$ and then compute the $\mathcal{I}_\sigma$ (see algorithm [CL98, 3.2]) and eventually compute the α_σ and β_σ.

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