Divisibility of \Sigma_k(N) by Even Perfect Numbers

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December 11, 2022

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Abstract

Let $n = 2^{\alpha-1}p^{\beta-1}$ be a positive integer, where $\alpha, \beta > 1$ and p is a prime satisfying $p < 2^{(\alpha-1-\frac{\ln\alpha}{\ln 2})} - 1$. Let $k > 2$ be a prime such that $2^k - 1$ is a Mersenne prime and $\sigma_k(n) = \sum_{d|n} d^k$ be the sum of the k^{th} power of positive divisors of n. Continuing the work of Chu [4], we prove that n divides $\sigma_k(n)$ if and only if and only n is an even perfect number$\neq 2^{k-1}(2^k - 1)$ for all $k \leqslant 31$.

1 Introduction and Main Results

A positive integer n is said to be perfect number if it satisfies the equation $\sigma(n) = 2n$. The study of perfect numbers dates back to antiquity with Euclid being the first to prove a notable result that every number of the form $n = 2^{\alpha-1}p$, where $p = 2^\alpha - 1$ is a Mersenne prime is an even perfect number. Euler later proved that an even integer n is perfect if and only if it is of the form $n = 2^{\alpha-1}p$, where $p = 2^\alpha - 1$ is a Mersenne prime. Only 51 even perfect numbers have ever since been discovered and it still remains unknown whether infinitely many even perfect numbers exist. It is

2020 *Mathematics Subject Classification*: 11A05 .
Key words and phrases: perfect numbers, divisor function.

also unknown whether any odd perfect numbers exist, though computations suggest that the smallest odd perfect number should be greater than 10^{1500} [9] and have at least 9 distinct prime divisors [8]. The study of perfect numbers has been generalized in various forms, noticeably in the context of k-perfect numbers, which are defined as integers n satisfying the relation $\sigma(n) = kn$ for some integer $k > 2$. Numerous results on k-perfect numbers have been established and we refer the reader here [10, 3] for related results. Mathematicians have continued to establish meaningful relations between a positive integer n and its k^{th} divisor function $\sigma_k(n) := \sum_{d|n} d^k$, often resulting in beautiful mathematical results. For instance, Cai et al [1] proved that n is a solution to the equation $\sigma_2(n) - n^2 = 3n$ if and only if n is a product of two Fibonacci primes. Cai et al [2] proved that infinitely many twin primes exists if and only if the equation $\sigma_2(n) - n^2 = 2n + 5$ has infinitely many solutions. Besides the relation of equality, generalizations for divisibility of $\sigma_k(n)$ by n have been considered as well. Florian et al [7] proved that n divides $\sigma_k(n)$ infinitely often for any $k \geqslant 2$. Divisibility of $\sigma_k(n)$ by n is particularly of interest when $n = 2^{\alpha-1}p^{\beta-1}$, where $\alpha, \beta \in \mathbb{Z}^+$ and p is a prime. In such a case, the divisibility of $\sigma_k(n)$ by n for $k = 3$ or 5 has been known to occur if and only if n is a perfect, provided $p < 3 \cdot 2^{\alpha-1} - 1$. Jiang [6] considered the case $k = 3$ and proved the following

Theorem 1.1. . *Let $n = 2^{\alpha-1}p^{\beta-1}$, where $\alpha, \beta > 1$ and p is an odd prime. Then $n|\sigma_3(n)$ if and only if n is an even perfect number$\neq 28$.*

Chu [4] recently noticed that Theorem 1.1 could not be extended to $k =$ 5 or 7 since there are non-perfect values of n for which $n|\sigma_k(n)$ in those cases. For example $\sigma_5(22) \equiv 0 \pmod{22}$ and $\sigma_7(86) \equiv 0 \pmod{86}$ even though 22 and 86 are not perfect. However by restricting p to satisfy the inequality $p < 3 \cdot 2^{\alpha-1} - 1$, Chu [4] proved the following Theorem for the case $k = 5$.

Theorem 1.2. *Let $n = 2^{\alpha-1}p^{\beta-1}$, where $\alpha, \beta > 1$ and $p < 3 \cdot 2^{\alpha-1} - 1$ is an odd prime. Then $n|\sigma_5(n)$ if and only if n is an even perfect number$\neq 496$.*

Chu [4] made a generalization of Theorem 1.2 in form of the following Conjecture.

Conjecture 1.3. . Let $k > 2$ be a prime such that $2^k - 1$ is a Mersenne prime.If $n = 2^{\alpha-1}p^{\beta-1}$, where $\alpha, \beta > 1$ and $p < 3 \cdot 2^{\alpha-1} - 1$ is an odd prime, then $n|\sigma_k(n)$ if and only if n is an even perfect number$\neq 2^{k-1}(2^k - 1)$.

Interestingly, Chu [4] proved that Conjecture 1.3 holds when $\beta = 2$ as stated below.

Theorem 1.4. . *Let $k > 2$ be a prime such that $2^k - 1$ is a Mersenne prime. If $n = 2^{\alpha-1}p$, where $\alpha > 1$ and $p < 3 \cdot 2^{\alpha-1} - 1$ is an odd prime, then $n|\sigma_k(n)$ if and only if n is an even perfect number$\neq 2^{k-1}(2^k - 1)$.*

The technique used to prove Theorem 1.2 could not be extended to prove Conjecture 1.3 for $k > 5$. In this paper, we extend Chu's [4] results by proving Conjecture 1.3 for all $k \leqslant 31$ and $p < 2^{(\alpha-1-\frac{\ln\alpha}{\ln 2})} - 1$ as stated in the following theorem

Theorem 1.5. *Let $k \leqslant 31$ be an odd prime such that $2^k - 1$ is a Mersenne prime.If $n = 2^{\alpha-1}p^{\beta-1}$, where $\alpha, \beta > 1$ and $p < 2^{(\alpha-1-\frac{\ln\alpha}{\ln 2})} - 1$ is an odd prime, then $n|\sigma_k(n)$ if and only if n is an even perfect number$\neq 2^{k-1}(2^k - 1)$.*

We structure our paper in the following way. Section 1 provides an introduction on the subject of our investigation leading up to the statement of the main results. In Section 2, we outline the necessary results for proving our main result and finally prove our main results in Section 3.

2 Preliminaries

Lemma 2.1. *Let $k > 2$ be a prime such that $2^k - 1$ is a Mersenne prime. Let $n = 2^{\alpha-1}p^{\beta-1}$, where $\alpha, \beta > 1$ and p is an odd prime. If $n|\sigma_k(n)$, then β is even,*

$$2^{\alpha-1} \mid \left(\frac{p^{\beta k} - 1}{p^k - 1}\right)$$

and

$$p^{\beta-1} \mid \left(\frac{2^{\alpha k} - 1}{2^k - 1}\right).$$

Chu [4] provides an elementary proof of Lemma 2.1.

Lemma 2.2. *Let $n = 2^{\alpha-1}p^{\beta-1}$ be a positive integer, $2^v||\beta$ and $n|\sigma_k(n)$, where $\alpha, \beta > 1$ and p is an odd prime..*

(a) If $p \equiv 1 \pmod 4$, then $\alpha - 1 \leqslant v$.

(b) If $p \equiv 3 \pmod 4$ and $2^{s-1}||(p+1)$, then $\alpha \leqslant v + s - 1$.

See ([4] Lemma 8 and Lemma 17) for proofs of Lemma 2.2 (a) and (b) respectively.

Lemma 2.3. *Let $\Phi_m(q)$ be the m^{th} cyclotomic polynomial of q, where $m, q \in \mathbb{Z}^+$. Let p be a prime divisor of $\Phi_m(2)$ such that $p^t|\Phi_m(2)$ for some $t \in \mathbb{Z}^+$. Then $t \leqslant 2$ for all $p < 6.75 \times 10^{15}$.*

See [5] for proof of Lemma 2.3

Lemma 2.4. *Let k, s, p, α be positive integers such that $2^{s-1}||(p+1)$ and $p < 2^{(\alpha-1-\frac{\ln\alpha}{\ln 2})} - 1$. Then the following inequalities hold.*

(a) $2^{\alpha-1} - 1 > \alpha k$ for all $\alpha > k + 2$.

(b) $2^{\alpha-s} > \alpha$.

Proof. (a) We have

$$2^{\alpha-1} - 1 = \sum_{i=0}^{k-1} {}^{\alpha-1}C_i > {}^{\alpha-1}C_2 + {}^{\alpha-1}C_{\alpha-3} = (\alpha-1)(\alpha-2) \tag{2.1}$$

Since the sequence $\left\{\dfrac{k}{k+1}\right\}_{k=1}^{\infty}$ is strictly increasing, it follows that

$$\frac{(k+t+1)}{(k+t+2)} > \frac{k}{(k+t)} \tag{2.2}$$

for all $t \geqslant 1$. Inequality (2.2) can be written as

$$(k+t+1)(k+t) > (k+t+2)k \tag{2.3}$$

Substituting $\alpha = k+t+2$ into (2.3) yields

$$(\alpha-1)(\alpha-2) > \alpha k \tag{2.4}$$

Combining inequalities (2.4) with (2.1) yields $2^{\alpha-1} - 1 > \alpha k$.

(b) Since $2^{s-1}||(p+1)$, we have $2^{s-1} - 1 \leqslant p < 2^{(\alpha-1-\frac{\ln\alpha}{\ln 2})} - 1$. It follows that $2^{s-1} < 2^{(\alpha-1-\frac{\ln\alpha}{\ln 2})}$, or equivalently $2^{\alpha-s} > \alpha$.

$\square$

3 Proof of Theorem 1.5

We start by proving the forward direction, that is if n is an even perfect number$\neq 2^{k-1}(2^k-1)$, then n divides $\sigma_k(n)$. Suppose n is perfect. Then $n = 2^{\alpha-1}p$ where α and $p = 2^\alpha - 1$ are primes. It immediately follows from the power of p in n that $\beta - 1 = 1$. To show that $n|\sigma_k(n)$, we need to show that

$2^{\alpha-1} \mid \Big(\dfrac{p^{\beta k}-1}{p^k-1}\Big)$ and $p^{\beta-1} \mid \Big(\dfrac{2^{\alpha k}-1}{2^k-1}\Big)$.

We notice from the binomial expansion

$$2^{\alpha k} - 1 = (2^\alpha - 1)\sum_{i=0}^{k-1} {}^kC_i(2^\alpha-1)^{k-i-1}$$

that $(2^\alpha - 1)$ divides $2^{\alpha k} - 1$. Since $n \neq 2^{k-1}(2^k-1)$, then $p \neq 2^k - 1$ and since both $2^\alpha - 1$ and $2^k - 1$ are primes, $\gcd(2^\alpha - 1, 2^k - 1) = 1$. Since $p = (2^\alpha - 1)$ divides $2^{\alpha k} - 1$, $(2^k - 1) \nmid (2^\alpha - 1)$ and $\beta - 1 = 1$, it follows that $p^{\beta-1} \mid \Big(\dfrac{2^{\alpha k}-1}{2^k-1}\Big)$.

It remains to show that $2^{\alpha-1}$ divides $\Big(\dfrac{p^{\beta k}-1}{p^k-1}\Big)$. Since $\beta - 1 = 1$ and $p = 2^\alpha - 1$, we notice from the binomial expansion

$$\left(\frac{p^{\beta k}-1}{p^k-1}\right)=\left(\frac{p^{2k}-1}{p^k-1}\right)=1+(2^\alpha-1)^k=2^\alpha\sum_{i=0}^{k-1}{}^kC_i(2^\alpha)^{k-i-1}(-1)^i.$$

that $2^{\alpha-1}$ divides $\left(\frac{p^{\beta k}-1}{p^k-1}\right)$.

Now we turn our attention to proving that if $n=2^{\alpha-1}p^{\beta-1}$ and $n|\sigma_k(n)$, then n is an even perfect number$\neq 2^{k-1}(2^k-1)$. We start by noting that the case $\beta=2$ is the case of Theorem 1.4 and has been fully addressed. Thus we only prove the case $\beta>2$.

We proceed by considering the cases $p\equiv 1 \pmod 4$ and $p\equiv 3 \pmod 4$ separately.

3.1 The case $p\equiv 1 \pmod 4$

Proposition 3.1. Let $n=2^{\alpha-1}p^{\beta-1}$, where $\alpha>1,\beta>2$ are integers and $p\equiv 1 \pmod 4$ is prime satisfying $p<3\cdot 2^{\alpha-1}-1$. Let $k\leqslant 31$ be an odd prime such that 2^k-1 is a Mersenne prime. Then $n\nmid\sigma_k(n)$.

Proof. We proceed by contradiction. Suppose $n\mid\sigma_k(n)$, then it follows from Lemma 2.1 that $p^{\beta-1}\mid\left(\frac{2^{\alpha k}-1}{2^k-1}\right)$. Suppose $\beta=2^v\beta_1$, where $\gcd(2,\beta_1)=1$. It follows from Lemma 2.2(a) that $\alpha-1\leqslant v$, from which we get $2^v-1\geqslant 2^{\alpha-1}-1$. It follows that

$$p^{\beta-1}\geqslant p^{2^v-1}\geqslant p^{2^{\alpha-1}-1} \tag{3.1}$$

If $\alpha>k+2$, it follows from Lemma 2.4(a) that $2^{\alpha-1}-1>\alpha k$, from which we get

$$p^{2^{\alpha-1}-1}>p^{\alpha k}\geqslant 5^{\alpha k}>\left(\frac{2^{\alpha k}-1}{2^k-1}\right) \tag{3.2}$$

The second inequality in (3.2) follows from the fact that $p\equiv 1 \pmod 4$. Joining inequalities (3.1) and (3.2) yields $p^{\beta-1}>\left(\frac{2^{\alpha k}-1}{2^k-1}\right)$ which is a contradiction.

Thus we must have $\alpha \leqslant k+2$. Since $k \leqslant 31$, we have $\alpha \leqslant 33$. By hypothesis, we have $p < 3 \cdot 2^{\alpha-1} - 1$, which together with $\alpha \leqslant 33$ yields

$$p < 3 \cdot 2^{\alpha-1} - 1 \leqslant 3 \cdot 2^{32} - 1 < 1.29 \times 10^{10}. \tag{3.3}$$

Inequality (3.3) implies $p < 6.75 \times 10^{15}$. By Lemma 2.3, we have $\beta - 1 \leqslant 2$ or $\beta \leqslant 3$. Since β is even and $\beta > 1$, we have $\beta = 2$ which is a contradiction. □

3.2 The case $p \equiv 3 \pmod 4$

Proposition 3.2. Let $n = 2^{\alpha-1}p^{\beta-1}$ where $p \equiv 3 \pmod 4$ is prime, $2^{s-1}||(p+1)$, $p < 2^{(\alpha-1-\frac{\ln\alpha}{\ln 2})}$ and $\alpha > 1, \beta > 2, s \geqslant 2$ are positive integers. Let $k \leqslant 31$ be an odd prime such that $2^k - 1$ is a Mersenne prime. Then $n \nmid \sigma_k(n)$.

Proof. We proceed by contradiction. Suppose $n \mid \sigma_k(n)$, then it follows from Lemma 2.1 that $p^{\beta-1} \mid \left(\dfrac{2^{\alpha k}-1}{2^k-1}\right)$.

Suppose $\beta = 2^v\beta_1$, where $\gcd(2, \beta_1) = 1$. Since $2^{s-1}||(p+1)$, it follows from Lemma 2.2(b) that $\alpha - s \leqslant v - 1$, from which we get $2^{v-1} \geqslant 2^{\alpha-s}$. It follows that

$$p^{\beta-1} > p^{2^{v-1}} \geqslant p^{2^{\alpha-s}} > 2^{(s-2)(2^{\alpha-s})} \tag{3.4}$$

Since $p < 2^{(\alpha-1-\frac{\ln\alpha}{\ln 2})}$ and $2^{s-1}||(p+1)$, it follows from Lemma 2.4(b) that $2^{\alpha-s} > \alpha$.

If $s-2 > k$, then $(s-2)(2^{\alpha-s}) > \alpha k$ and it follows that

$$2^{(s-2)(2^{\alpha-s})} > 2^{\alpha k} > \left(\frac{2^{\alpha k}-1}{2^k-1}\right) \tag{3.5}$$

Joining inequalities (3.4) and (3.5) yields $p^{\beta-1} > \left(\dfrac{2^{\alpha k}-1}{2^k-1}\right)$ which is a contradiction.

If $s-2 \leqslant k$, then $\alpha - s \geqslant \alpha - k - 2$. Thus $2^{\alpha-s} \geqslant 2^{\alpha-k-2}$, from which it follows that

$$2^{(s-2)(2^{\alpha-s})} > 2^{2^{\alpha-s}} \geqslant 2^{2^{\alpha-k-2}} \tag{3.6}$$

Of all the values $k \leqslant 31$, $2^k - 1$ is prime for k values $3, 5, 7, 13, 17, 19$ and 31. It can be proved by induction that the inequality

$$2^{\alpha-k-2} > \alpha k \tag{3.7}$$

holds for all $k \leqslant 31$ and $\alpha \geqslant 44$. Joining inequalities (3.4), (3.6) and (3.7) yields

$$p^{\beta-1} > 2^{2^{\alpha-k-2}} > 2^{\alpha k} > \left(\frac{2^{\alpha k}-1}{2^k-1}\right) \text{ for all } \alpha \geqslant 44$$

which is a contradiction.
For the case $\alpha < 44$, it follows from $p < 2^{(\alpha - 1 - \frac{\ln \alpha}{\ln 2})} - 1$ that $p < 2^{38} < 6.75 \times 10^{15}$. By Lemma 2.3, it follows that $\beta - 1 \leqslant 2$ or $\beta \leqslant 3$. Since β is even and $\beta > 1$, we have $\beta = 2$ which is a contradiction.

□

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