Riemann Hypothesis on Grönwall's Function

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Abstract

Grönwall's function G is defined for all natural numbers $n > 1$ by $G(n) = \frac{\sigma(n)}{n \cdot \log \log n}$ where $\sigma(n)$ is the sum of the divisors of n and log is the natural logarithm. We require the properties of extremely abundant numbers, that is to say left to right maxima of $n \mapsto G(n)$. We also use the colossally abundant and hyper abundant numbers. There are several statements equivalent to the famous Riemann hypothesis. It is known that the Riemann hypothesis is true if and only if there exist infinitely many extremely abundant numbers. In this note, using this criterion on hyper abundant numbers, we prove that the Riemann hypothesis is true.

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1 Introduction

As usual $\sigma(n)$ is the sum-of-divisors function of n

$$\sum_{d \mid n} d,$$

where $d \mid n$ means the integer d divides n. In 1997, Ramanujan's old notes were published where it was defined the generalized highly composite numbers, which include the superabundant and colossally abundant numbers [6]. A natural number n is called superabundant precisely when, for all natural numbers $m < n$

$$\frac{\sigma(m)}{m} < \frac{\sigma(n)}{n}.$$

A number n is said to be colossally abundant if, for some $\epsilon > 0$,

$$\frac{\sigma(n)}{n^{1+\epsilon}} \geq \frac{\sigma(m)}{m^{1+\epsilon}} \ \ for \ \ (m > 1).$$

Every colossally abundant number is superabundant [1]. Let us call hyper abundant an integer n for which there exists $u > 0$ such that

$$\frac{\sigma(n)}{n \cdot (\log n)^u} \geq \frac{\sigma(m)}{m \cdot (\log m)^u} \quad \textit{for} \quad (m > 1),$$

where log is the natural logarithm. Every hyper abundant number is colossally abundant [5, pp. 255]. In 1913, Grönwall studied the function $G(n) = \frac{\sigma(n)}{n \cdot \log \log n}$ for all natural numbers $n > 1$, [3]. We have the Grönwall's Theorem:

Proposition 1.1.

$$\limsup_{n \to \infty} G(n) = e^{\gamma}$$

where $\gamma \approx 0.57721$ *is the Euler-Mascheroni constant [3].*

Next, we have the Robin's Theorem:

Proposition 1.2. *The Riemann hypothesis is true if and only if* $G(n) < e^{\gamma}$ *for every natural number* $n > 5040$ *[7, Theorem 1 pp. 188].*

There are champion numbers (i.e. left to right maxima) of the function $n \mapsto G(n)$:

$$G(m) < G(n)$$

for all natural numbers $10080 \leq m < n$. A positive integer n is extremely abundant if either $n = 10080$, or $n > 10080$ is a champion number of the function $n \mapsto G(n)$. In 1859, Bernhard Riemann proposed his hypothesis [2]. Several analogues of the Riemann hypothesis have already been proved [2].

Proposition 1.3. *The Riemann hypothesis is true if and only if there exist infinitely many extremely abundant numbers [4, Theorem 7 pp. 6].*

Many authors expect (or at least hope) that it is true. However, there are some implications in case of the Riemann hypothesis might be false.

Proposition 1.4. *If the Riemann hypothesis is false then, for colossally abundant numbers* N *we have*

$$G(N) = e^{\gamma} \cdot \left(1 + \Omega_{\pm}((\log N)^{-b})\right)$$

for some $0 < b < 1$ *[7, Proposition pp. 204].*

This is our main theorem

Theorem 1.5. *The Riemann hypothesis is true.*

Putting all together yields a proof for the Riemann hypothesis using the hyper abundant numbers.

2 Central Lemma

Lemma 2.1. *For two real numbers $y > x > e$:*

$$\frac{y}{x} > \frac{\log y}{\log x}.$$

Proof. We have $y = x + \varepsilon$ for $\varepsilon > 0$. We obtain that

$$\begin{aligned}\frac{\log y}{\log x} &= \frac{\log(x+\varepsilon)}{\log x} \\ &= \frac{\log\left(x \cdot (1 + \frac{\varepsilon}{x})\right)}{\log x} \\ &= \frac{\log x + \log(1 + \frac{\varepsilon}{x})}{\log x} \\ &= 1 + \frac{\log(1 + \frac{\varepsilon}{x})}{\log x}\end{aligned}$$

and

$$\begin{aligned}\frac{y}{x} &= \frac{x+\varepsilon}{x} \\ &= 1 + \frac{\varepsilon}{x}.\end{aligned}$$

We need to show that

$$\left(1 + \frac{\log(1 + \frac{\varepsilon}{x})}{\log x}\right) < \left(1 + \frac{\varepsilon}{x}\right)$$

which is equivalent to

$$\left(1 + \frac{\varepsilon}{x \cdot \log x}\right) < \left(1 + \frac{\varepsilon}{x}\right)$$

using the well-known inequality $\log(1 + x) \le x$ for $x > 0$. For $x > e$, we have

$$\frac{\varepsilon}{x} > \frac{\varepsilon}{x \cdot \log x}.$$

In conclusion, the inequality

$$\frac{y}{x} > \frac{\log y}{\log x}$$

holds on condition that $y > x > e$. □

3 Main Insight

Lemma 3.1. *Every large enough hyper abundant number n is defined over a parameter $1 > u \gtrapprox 0$.*

Proof. Every large enough hyper abundant number n is defined over a parameter $u > 0$ as follows,

$$\frac{\sigma(n)}{n \cdot (\log n)^u} \geq \frac{\sigma(m)}{m \cdot (\log m)^u} \quad \textit{for} \quad (m > 1).$$

Then, we would have

$$\frac{\sigma(n)}{n \cdot (\log n)^u} \geq \frac{\sigma(n')}{n' \cdot (\log n')^u} \quad \textit{for} \quad (n > n' > e).$$

Thus,

$$\left(\frac{\log n'}{\log n}\right)^u \geq \frac{\frac{\sigma(n')}{n'}}{\frac{\sigma(n)}{n}}.$$

By Proposition 1.1, we know there exists some $n' \leq 5040$ such that

$$\frac{\frac{\sigma(n')}{n'}}{\frac{\sigma(n)}{n}} \geq \frac{\log \log n'}{\log \log n}$$

for large enough hyper abundant number n. Hence, we would obtain that

$$\left(\frac{\log n'}{\log n}\right)^u \geq \frac{\log \log n'}{\log \log n}.$$

Moreover, we know that

$$\left(\frac{\log \log n'}{\log \log n}\right)^u > \left(\frac{\log n'}{\log n}\right)^u$$

since $n > n' > e$ by Lemma 2.1. However, this implies that

$$\left(\frac{\log \log n'}{\log \log n}\right)^u > \frac{\log \log n'}{\log \log n}.$$

which immediately forces the parameter u to be necessarily lesser than 1. Now, we will show that $u \gtrapprox 0$. Consider there is pair (n, n') of two consecutive hyper abundant numbers such that $n < n'$ and they are defined over the parameters u and u', respectively. By definition of hyper abundant numbers, we have

$$\frac{\sigma(n)}{n \cdot (\log n)^u} \geq \frac{\sigma(n')}{n' \cdot (\log n')^u}$$

and

$$\frac{\sigma(n')}{n' \cdot (\log n')^{u'}} \geq \frac{\sigma(n)}{n \cdot (\log n)^{u'}}.$$

That would mean

$$\left(\frac{\log n'}{\log n}\right)^{u} \geq \frac{\frac{\sigma(n')}{n'}}{\frac{\sigma(n)}{n}} \geq \left(\frac{\log n'}{\log n}\right)^{u'}$$

and therefore, we obtain that $u \geq u'$ which implies that u decreases as n increases where this means that u tends to 0 as n goes to infinity and thus, $u \gtrapprox 0$. □

4 Proof of Theorem 1.5

Proof. Suppose that the Riemann hypothesis is false. Thus, there are not infinitely many extremely abundant numbers by Proposition 1.3. Then for Propositions 1.2 and 1.4 we infer that the maximum

$$M = max\{G(n) : n > 5040\}$$

exists and that $M > e^{\gamma}$. Besides, there is only a finite set of natural numbers $n > 5040$ such that $G(n) = M$ by Proposition 1.1 and the properties of limit superior. Certainly, suppose there would be an infinite increasing subsequence of natural numbers $n_i > 5040$ such that $e^{\gamma} < M = G(n_i)$. By definition of limit superior, for any positive real number ε, only a finite number of elements of the sequence $G(n)$ are greater than $e^{\gamma} + \varepsilon$ over all natural numbers $n > 1$ which is a contradiction with the fact that $G(n_i) = M$ and $e^{\gamma} + \varepsilon < M$ for all i. Since the set of natural numbers $n > 5040$ such that $G(n) = M$ is finite, then there must exist a maximum number N in this set.

We consider a large enough colossally abundant number N' such that $N < N'$. Let's assume that N' is a hyper abundant number with a parameter $u > 0$. This is possible since every hyper abundant number is colossally abundant [5, pp. 255]. Under our assumption, we have

$$\frac{\sigma(N)}{N \cdot (\log \log N)} > \frac{\sigma(N')}{N' \cdot (\log \log N')}$$

which is

$$\frac{\sigma(N)}{N \cdot (\log N)^{u'}} > \frac{\sigma(N')}{N' \cdot (\log \log N')}$$

where

$$\log \log N = (\log N)^{u'}.$$

We know the parameter $1 > u \gtrapprox 0$ tends to be smaller as long as N' become into a larger hyper abundant number by Lemma 3.1. In this way, we obtain that $u' \gg u$ where $\gg$ means "much greater than". Consequently,

$$\frac{\sigma(N)}{N \cdot (\log N)^u} > \frac{\sigma(N)}{N \cdot (\log N)^{u'}} = \frac{\sigma(N)}{N \cdot (\log \log N)}.$$

By definition of hyper abundant numbers, we have

$$\frac{\sigma(N')}{N' \cdot (\log N')^u} \geq \frac{\sigma(N)}{N \cdot (\log N)^u}.$$

So,

$$\frac{\sigma(N')}{N' \cdot (\log N')^u} \geq \frac{\sigma(N)}{N \cdot (\log N)^u} > \frac{\sigma(N)}{N \cdot (\log \log N)}$$

and therefore,

$$\log \log N > (\log N')^u.$$

There are infinitely many hyper abundant numbers since for all $u > 0$:

$$\lim_{n \to \infty} \frac{\sigma(n)}{n \cdot (\log n)^u} = 0$$

and thus, $\frac{\sigma(n)}{n \cdot (\log n)^u}$ is bounded [5, pp. 254–255]. Hence, when N' ranges over the set of large enough hyper abundant numbers:

$$\gamma < \log\left(\frac{\sigma(N)}{N}\right) - \log \log \log N < \log\left(\frac{\sigma(N)}{N}\right) - u \cdot \log \log N' \approx 0$$

when the Riemann hypothesis is false. However, we know that

$$0 \gtrapprox \gamma$$

is trivially false and thus, we obtain a contradiction just assuming that the Riemann hypothesis is false. By reductio ad absurdum, we deduce that the Riemann hypothesis is indeed true. $\square$

5 Conclusions

Practical uses of the Riemann hypothesis include many propositions that are known to be true under the Riemann hypothesis and some that can be shown to be equivalent to the Riemann hypothesis. Indeed, the Riemann hypothesis is closely related to various mathematical topics such as the distribution of primes, the growth of arithmetic functions, the Lindelöf hypothesis, the Large Prime Gap Conjecture, etc. Certainly, a proof of the Riemann hypothesis could spur considerable advances in many mathematical areas, such as number theory and pure mathematics in general.

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