Queueing Theory: Rouche's Theorem: Localization Theorem

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ABSTRACT

Traditionally, in the analysis of queueing models based on complex analysis method and the associated Skip Free Markov chain based models, Rouche's Theorem is invoked to determine the eigenvalues of the recursion matrix, called rate matrix which is the solution of a matrix polynomial/ matrix power series equation. In this research paper, we provide a simple proof of the Localization Theorem using the linear algebraic arguments. We also, prove a generalized Perron Theorem associated with a polynomial matrix

1. INTRODUCTION:

Mathematical modeling of systems evolving in time has proven to be very valuable in predicting the performance of dynamical systems. Specifically, stochastic (non-deterministic) dynamical systems naturally arise in many interesting applications. Following the principle of Occam's Razor, Continuous Time Markov Chains (CTMCs) are widely used in analyzing the performance of many stochastic dynamical systems. It was well established that such Markov chains exhibit equilibrium probability distribution (associated with the states in the state space). Efficient computation of such equilibrium probability distribution is an ever green research problem. For instance, in the case of a birth-and-death process (a special type of CTMC), the equilibrium probability distribution is specified by a geometric sequence. Thus, the common-ratio, ρ (of the geometric sequence) as well as the probability of the initial state will completely specify the equilibrium probability distribution. The common ratio, ρ is the smallest zero/root of the following scalar quadratic equation:

$$x^2\beta - x(\alpha+\beta) + \alpha = 0 \quad with\ smallest\ zero\ \rho = \frac{\alpha}{\beta}.$$

In the above equation, α is the birth rate and β is the death rate. Also '1' is the largest zero of the scalar quadratic equation.

As a natural generalization of birth-and-death process, Quasi-Birth-and-Death process is Proposed in which the state space is partitioned into "levels" that are groups of finitely many states. As in the birth-and-death process, state transitions take place only to the adjacent (higher, lower) levels. Furthermore, the equilibrium distribution has matrix-geometric recursive form i.e.

$$\bar{\pi}(n+1) = \bar{\pi}(n)\, R \,, where \text{ ………………………………}(1)$$

$\bar{\pi}(n)$ is the row vector of equilibrium probabilities of states at level 'n' and R is called the rate matrix which is the minimal non-negative solution of the matrix quadratic equation

$$R^2 A_2 + R A_1 + A_0 \equiv \bar{0} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (2).$$

In the above matrix equation, $A_i's$ are the state transition rate matrices.

For the vector valued infinite sequence in (1) to converge, it is necessary that all the eigenvalues of rate matrix, R should lie strictly within the unit circle. As proved in Section (3), all the eigenvalues of every matrix solution, $\bar{X}$ of the matrix quadratic equation

$$X^2 A_2 + X A_1 + A_0 \equiv \bar{0} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (3)$$

are the zeroes/roots of the determinental polynomial

$$Det(\mu^2 A_2 + \mu A_1 + A_0) = f(\mu) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (4).$$

Thus, as reasoned in Section (2), there should be only 'N' zeroes of $f(\mu)$ which lie within the unit circle and all other zeroes are on or outside the unit circle. As discussed in detail in Section (2), traditional proof that there are only N zeroes of $f(\mu)$ (in equation (4)) which lie strictly within the unit circle is by the so called "Complex Analysis" method. The method involves invoking the ROUCHE's THEOREM from complex Analysis. In the PhD thesis [1], the author explored an alternative proof of the fact without invoking Rouche's theorem. This research paper provides a much simpler proof without invoking the Rouche's theorem (using linear algebraic argument only).

This research paper is organized as follows. In Section 2, the known research literature is briefly reviewed. In Section 3, localization Theorem associated with the matrix polynomial equation satisfied by the rate matrix is proved. Also, an interesting generalized Perron theorem is proved. The research paper concludes in Section 4.

2. Matrix Analytic Methods in Queueing Theory: Rouche's Theorem:

Queueing theorists proposed models of queueing systems which are in the most general case, generalizations of Quasi-Birth-and-Death (QBD) process. Those queueing systems are modeled as a G/M/1-type Markov chain. The generator matrix of such an infinite state space CTMC is of the block Toeplitz form

The equilibrium PMF vector (infinite dimensional) of such a CTMC has the associated Matrix Geometric Recursion

$$\bar{\pi}(n+1) = \bar{\pi}(n)\, R \quad for \ n \geq 0, \ where \ \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$\bar{\pi}(n)$ is the row vector of equilibrium probabilities of states at level 'n' and R is the minimal nonnegative solution of the matrix power series equation

$$\sum_{i=0}^{\infty} R^i A_i \equiv \bar{0}, where \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$A_i's$ are the submatrices of the generator matrix (infinite dimensional) corresponding to the state transition rates on level 'n'. It readily follows that the eigenvalues are necessarily the zeroes of the transcendental function

$$h(\mu) = \ Det \ \left(\sum_{i=0}^{\infty} \mu^i A_i \right). \text{...} (3)$$

This result follows from a generalization of Factorization Lemma in [1]. In the Complex analysis based approaches (in queueing theory), Rouche's Theorem is Invoked to prove that $h(\mu)$ has exactly 'N' zeroes that are the roots (of the Transcendental function) of interest in determination of the equilibrium probabilities. From complex analysis, Rouche's theorem leads to the following result:

If f(z) and g(z) are two analytic functions within and on a simple closed curve C such that |f(z)| > |g(z)| at each point on C, then both f(z) and f(z) + g(z) have the same number of zeros inside C.

In [1], for the first time, a NOVEL PROOF of the fact that $h(\mu)$ has N zeroes strictly inside the unit circle was provided based on linear algebraic arguments. The result was coined as a "Localization Theorem". In this research paper, we provide a simpler proof of localization theorem without invoking the Rouche's Theorem. Most results in queueing theory based on Rouche's Theorem could be interpreted/arrived at by the Localization Theorem.

3. Matrix Polynomial Equations: Localization Theorem:

Consider a G/M/1-type Markov Chain in which the states at any level can only receive downward transitions from the states that are only finitely many (specifically 'L' levels) levels up. In such a case, the rate matrix is a solution of the following matrix polynomial equation i.e.

$$\sum_{i=0}^{L} R^i A_i \equiv \bar{0} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (4)$$

The following factorization lemma enables characterization of eigenvalues of rate matrix R

Lemma 1: Consider a matrix polynomial equation in unknown matrix X and coefficient matrices $B_i's$ i.e. $\sum_{i=0}^{L} X^i B_i \equiv \bar{0}$. We readily have the following factorization

$$G(\mu) = \sum_{i=0}^{L} \mu^i B_i \equiv (\mu \ I - X)\,(\mu^{L-1} C_1 + \ \mu^{L-2} C_2 + \cdots + \mu C_{L-1} + C_L), \qquad where$$

$$C_1 = \ B_L, \qquad C_2 = B_{L-1} + X\, B_L, \ \ C_3 = \ B_{L-2} + X\, B_{L-1} + X^2 B_L \ldots\ldots\ldots\ldots.$$

$$C_{L-1} = B_2 + X\, B_3 + \cdots + X^{L-2} B_L \quad and$$

$$C_L = B_1 + X\, B_2 + \cdots + X^{L-1}\, B_L$$

Proof: Refer [2]

Thus, the above Lemma applies to the matrix polynomial equation satisfied by the rate matrix R. Using the factorization Lemma, a Localization Theorem associated with the eigenvalues of R was first proved in [1].

LOCALIZATION THEOREM : Consider the matrix polynomial equation satisfied by the rate matrix, R i.e.

$$\sum_{i=0}^{L} R^i A_i \equiv \bar{0} \quad with$$

$$\sum_{i=0}^{L} \mu^i A_i \equiv (\mu \ I - X)\, H(\mu) = \ (\mu \ I - X)\,(\mu^{L-1} D_1 + \ \mu^{L-2} D_2 + \cdots + \mu D_{L-1} + D_L), where$$

$$D_1 = \ A_L, \qquad D_2 = A_{L-1} + R\, A_L, \ \ D_3 = \ A_{L-2} \ + R\, A_{L-1} + R^2 A_L \ldots\ldots\ldots\ldots\ldots.$$

$$D_{L-1} = A_2 \ + R\, A_3 + \cdots + R^{L-2} A_L \quad and$$

$$D_L = A_1 \ + R\, A_2 + \cdots + R^{L-1}\, A_L\, .$$

For $|\mu| < 1$, $H(\mu)$ is a non-singular matrix. Hence, all the zeroes of $Det\,(\sum_{i=0}^{L} \mu^i A_i)$ which lie within the unit circle are all eigenvalues of the rate matrix, R.

Proof: We now provide a new, simpler proof of the localization theorem dealing with the eigenvalues of rate matrix, R. From the factorization Lemma, we have that

$$\sum_{i=0}^{L} \mu^i A_i \equiv (\mu \ I - X)\, H(\mu), \ \ where \ \ D_i\}_{i=1}^{L-1} \quad are \ \ all \ \ non-negative \ \ matrices,$$

$$since, R \ \ is \ \ a \ \ non-negative \ \ matrix \ \ and \ \ A_i\}_{i=2}^{L} \ are \ also \ non-negative.$$

$$Further, \qquad the \ A_i^{'}s \ \ being \ matrices \ of \ the \ generator \ matrix \ on \ a \ level,$$

$$we \ \ have \ \ that \ \ \left(\sum_{i=0}^{L} A_i\right) \bar{e} \ = \ \bar{0},$$

$$where \ \ \bar{e} = (\, 1\, 1 \ldots 1\,)^T \ i.e. \ a \ \ column\, vector \ \ of \ \ all \ \ ones.$$

Also, from the factorization lemma satisfied by R, for $\mu = 1$, $we \ \ have \ \ that$

$$\sum_{i=0}^{L} A_i = (\, I - R\,) \left(\sum_{i=1}^{L} D_i\right) \ \ and$$

$$\left(\sum_{i=0}^{L} A_i\right) \bar{e} = (\, I - R\,) \left(\sum_{i=1}^{L} D_i\right) \bar{e} \ = \ \bar{0}.$$

But, since the spectral radius of R is strictly less than one (for arriving at the equilibrium probability distribution from the matrix geometric recursion)

$$\left(\sum_{i=1}^{L} D_i\right) \bar{e} \ = \ \bar{0}.$$

Also, from the definition (in the factorization lemma applied to R), we have that

$$D_L = A_1 \ + R\, A_2 + \cdots + R^{L-1}\, A_L.$$

Thus, it readily follows that D_L has negative diagonal elements and non-negative off diagonal elements. Since, $D_i\}_{i=1}^{L-1}$ are all non-negative, it follows that, for $|\mu| < 1$

$$(\mu^{L-1} D_1 + \mu^{L-2} D_2 + \cdots + \mu D_{L-1} + D_L)$$

Is a diagonally dominant matrice with negative diagonal elements, non-negative off diagonal elements and all the row sums being strictly less than one.

Thus, for $|\mu| < 1$, $H(\mu)$ *is a non-singular matrix*. Hence, all the zeroes of $Det\ (\sum_{i=0}^{L} \mu^i A_i)$ which lie within the unit circle are all eigenvalues of the rate matrix, R. Q.E.D.

Note: The proof argument generalizes for the matrix power series equation satisfied by the rate matrix R (in the case of arbitrary G/M/1 type Markov chain). Details are avoided for brevity.

Based on the factorization lemma1 above, we now provide the following generalization of Perron's Theorem after stating the Perron's theorem for completeness

Perron's Theorem: Any real square matrix , J with positive entries has a unique eigenvalue of largest magnitude and that eigenvalue is real.

Note: Suppose, we consider $K(\mu) = Det(I - \mu J)$ (i.e. a polynomial related to the characteristic polynomial of positive matrix J). Then by Perron's Theorem, the smallest zero of $K(\mu)$ is positive and real.
The following is a generalization of such result to the polynomial matrix.

GENERALIZED PERRON THEOREM: Consider POSITIVE square matrices $R_i\}_{i=1}^{S}$.
Define the polynomial matrix

$$I - \mu R_1 - \mu^2 R_2 - \cdots - \mu^S R_S \ .$$

The determinant of such a polynomial matrix i.e. $Det\ (I - \mu R_1 - \mu^2 R_2 - \cdots - \mu^S R_S)$ always has the smallest unique zero being real and positive

Proof: The main idea is to consider the characteristic polynomial of the following Block Companion matrix

$$\tilde{F} = \begin{bmatrix} \bar{0} & \bar{I} & \bar{0} & \cdots & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{I} & \cdots & \bar{0} & \bar{0} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \bar{0} & \bar{0} & \bar{0} & \cdots & \bar{I} & \bar{0} \\ \overline{R_s} & \overline{R_{s-1}} & \overline{R_{s-2}} & \cdots & \overline{R_2} & \overline{R_1} \end{bmatrix}.$$

Based on well known results related to such a block matrix, it readily follows that

$$Det(I - \mu \tilde{F}) = Det\ (I - \mu R_1 - \mu^2 R_2 - \cdots - \mu^S R_S)\ .$$

Thus, using the Perron's Theorem applied to the irreducible, non-negative matrix, $\tilde{F}$ we infer that the smallest zero of such determinental polynomial is real and positive. Q.E.D.

Note: As in the case of Perron-Frobenius Theorem (a generalization of Perron's Theorem), a generalized Perron-Frobenius Theorem can easily be proved. It is avoided for brevity. Also, without using Block companion matrix, Perron's argument can be generalized. Detailed argument is avoided for brevity.

We now illustrate the utilization of "generalized Perron theorem" in the context of the rate matrix based polynomial matrix.

Based on the proof of localization theorem, the matrix D_L is a diagonally dominant matrix with all diagonal elements being strictly negative. Hence, the inverse of it exists and it readily follows that the inverse is a non-positive matrix. Hence, the matrices, $R_j = -D_L^{-1} D_{L-j}$ is a positive/non-negative matrix for all $1 \leq j \leq (L-1)$. Thus, the polynomial matrix $Det\,(I - \mu R_1 - \mu^2 R_2 - \cdots - \mu^S R_S)$ has the smallest zero which is real and positive and it equals '1' in this case.

4. CONCLUSIONS:

In this research paper, based on purely linear algebraic arguments, an interesting Localization Theorem to determine the eigenvalues of rate matrix arising in equilibrium analysis of Skip Free Markov chains is proved. This approach can potentially provide an alternative to Rouche's Theorem based proof utilized in complex analysis based methods in queueing theory. We also prove a generalization of Perron's Theorem.

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