Self-Extensionality of Finitely-Valued Logics: Advances

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Abstract. We start from proving general characterizations of both self-extensionality and structural completeness of sentential logics as well as admissibility of rules in them, implying the decidability of these problems for (not necessarily uniform) finitely-valued logics. And what is more, in case of logics defined by finitely many either implicative or both disjunctive and conjunctive finite *hereditarily* simple (viz., having no non-simple submatrix) matrices, we then derive a characterization of self-extensionality yielding a quite effective algebraic criterion of checking their self-extensionality via analyzing homomorphisms between (viz., in the uniform case, endomorphisms of) the underlying algebras of their defining matrices and equally being a quite useful heuristic tool, manual applications of which are demonstrated within the framework of Łukasiewicz' finitely-valued logics, unform three-valued logics with subclassical negation (U3VLSN), uniform four-valued expansions of Belnap's "useful" four-valued logic as well as their (not necessarily uniform) no-more-than-four-valued extensions, [uniform inferentially consistent proper {in particular, no-more-than-three-valued} non-]classical ones proving to be [non-]self-extensional. Likewise, within the framework of classical (not necessarily functionally complete) logics and U3VLSN as well as uniform four-valued expansions of Belnap's logic, we obtain quite effective algebraic criteria of structural completeness.

1. Introduction

Perhaps, the principal value of universal logical investigations consists in discovering uniform points behind particular results originally proved *ad hoc*. This thesis is the main paradigm of the present universal logical study.

Recall that a sentential logic (cf., e.g., [7]) is said to be *self-extensional*, whenever its inter-derivability relation is a congruence of the formula algebra (i.e. is preserved under subformula replacement). This feature is typical of both two-valued (in particular, classical)[1] and super-intuitionistic logics as well as some interesting many-valued ones (like Belnap's "useful" four-valued one [2]). Here, we explore self-extensionality laying a special emphasis onto the general framework of finitely-valued logics and the decidability issue with reducing the complexity of effective procedures of verifying self-extensionality, when restricting our consideration to finitely-valued logics of special kind — namely, those defined by finitely many either implicative or both conjunctive and disjunctive (and so having either classical implication or both classical conjunction and classical disjunction in Tarski's conventional sense) *hereditarily simple* (viz., having no non-simple submatrix; i.e., having an *equality determinant* in a sense extending [18]) finite matrices.

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[1]Properly speaking, within the context of General Logic, the notorious classical logic arises just as the clone of miscellaneous functionally complete two-valued logics with classical negation. Here, we follow this natural paradigm, equally adopted in [23] even without the stipulation of functional completeness, calling functionally complete classical logics *genuinely* so, that naturally gives rise to the conception of *subclassical* logic/negation.

We then exemplify our universal elaboration by discussing four (perhaps, most representative) generic classes of logics of the kind involved: Łukasiewicz' finitely-valued logics [8]); unform three-valued logics with subclassical negation (U3VLSN); uniform four-valued expansions of Belnap's "useful" four-valued logic [2] as well as their (not necessarily uniform) no-more-than-four-valued extensions, [uniform inferentially consistent proper {in particular, no-more-than-three-valued} non-]classical ones proving to be [non-]self-extensional.

Likewise, a sentential calculus/logic is said to be *structurally complete*, whenever every rule, being *admissible in* it (i.e., retaining its *theorems* [viz., axioms derivable/satisfied in it]), is derivable/satisfied in it. Though the problem of verifying structural completeness of (not necessarily uniform) finitely-valued logics is decidable, its computational complexity is normally too large to apply it expansively. On the other hand, within the framework of classical (not necessarily functionally complete) logics and U3VLSN as well as uniform four-valued expansions of Belnap's logic, we obtain quite effective algebraic criteria of structural completeness.

The rest of the paper is as follows. The exposition of the material of the paper is entirely self-contained (of course, modulo very basic issues concerning Set and Lattice Theory, Universal Algebra and Logic to be found, if necessary, in standard mathematical handbooks like [1, 4, 10, 11]). Section 2 is a concise summary of particular basic issues underlying the paper, most of which, though having become a part of algebraic and logical folklore, are still recalled just for the exposition to be properly self-contained. Likewise, in Section 3, we then summarize certain advanced generic issues concerning simple matrices, equality determinants, intrinsic varieties as well as both disjunctivity and implicativity. Section 4 is a collection of main *general* results of the paper concerning self-extensionality that are then exemplified in Section 6 (aside from Łukasiewicz' finitely-valued logics, whose non-self-extensionality has actually been due [19], as we briefly discuss within Example 4.17 — this equally concerns certain particular instances discussed in Section 6 and summarized in Example 4.18). Likewise, in Section 5 we discuss the decidability of the issue of structural completeness and its computational complexity, advanced studying it within the framework of classical (not necessarily functionally complete) logics and U3VLSN as well as uniform four-valued expansions of Belnap's logic being presented in Section 6. Finally, Section 7 is a brief summary of principal contributions of the paper.

2. Basic issues

2.1. **Set-theoretical background.** We follow the standard set-theoretical convention (cf. [11]), according to which natural numbers (including 0) are treated as finite ordinals (viz., sets of lesser natural numbers), the ordinal of all them being denoted by ω. In this way, when dealing with n-tuples to be viewed as either [comma separated] sequences of length n or functions with domain n, where $n \in \omega$, π_i, where $i \in n$, denotes the i-th projection operator under enumeration started from rather 0 than 1. (In particular, when $n = 2$, $\pi_{0/1}$ denotes the left/right projection operator, respectively.) The proper class of all ordinals is denoted by ∞. Also, functions are viewed as binary relations (in particular, n-ary operations on a set A, where $n \in \omega$, are treated as $(n+1)$-ary relations on A), while singletons (viz., one-element sets) are identified with their unique elements, unless any confusion is possible. A function/mapping f /"to a set A" is said to be *singular/surjective*, provided $(\operatorname{img} f)$ is one-element/"equal to A", respectively.

Given a set S, let $\Delta_S \triangleq \{\langle a, a\rangle \mid a \in S\}$, relations of such a kind being referred to as *diagonal*, functions with diagonal kernel being said to be *injective*, "bijective" standing for "both injective and surjective", and $\wp_{[K]}(S)$ the set of all subsets of

S [of cardinality $\in K \subseteq \infty$], respectively. Then, given any *equivalence relation θ on S*, viz., a *transitive* (in the sense that $(\theta \circ \theta) \subseteq \theta$) *symmetric* (in the sense that $\theta^{-1} \subseteq \theta$) *reflexive binary relation on S* (in the sense that $\Delta_S \subseteq \theta \subseteq S^2$), ν_θ denotes the function with domain S defined by $\nu_\theta(a) \triangleq \theta[\{a\}]$, for all $a \in S$, while $(T/\theta) \triangleq \nu_\theta[T]$, for every $T \subseteq S$. Next, any *S-tuple* (viz., a function with domain S) is often written in the sequence form $\bar{t}$, its s-th component (viz., the value under argument s), where $s \in S$, being written as t_s, in that case. Given two more sets A and B, any relation $R \subseteq (A \times B)$ (in particular, a mapping $R : A \to B$) determines the equally-denoted relation $R \subseteq (A^S \times B^S)$ (resp., mapping $R : A^S \to B^S$) point-wise. Furthermore, any $f : S^n \to S$, where $n \in \omega$, is said to be *R-monotonic*, where $R \subseteq S^2$, provided, for all $\bar{a} \in R^n$, it holds that $\langle f(\bar{a} \circ \pi_0), f(\bar{a} \circ \pi_1)\rangle \in R$. Then, $\mathrm{Tr}(R) \triangleq \{\langle \pi_0(a_0), \pi_1(a_{m-1})\rangle \mid m \in (\omega \setminus 1), \bar{a} \in R^m, \forall i \in (m-1) : \pi_1(a_i) = \pi_0(a_{i+1})\}$ is the least transitive binary relation on S including R, called the *transitive closure of* R. Finally, given any $T \subseteq S$, we have the *characteristic function/mapping* $\chi^T_S \triangleq ((T \times \{1\}) \cup ((S \setminus T) \times \{0\})) \in 2^S$ *of T in S*.

Let A be a set. Then, a $U \subseteq \wp(A)$ is said to be *upward-directed*, provided, for every $S \in \wp_\omega(U)$, there is some $T \in U$ such that $(\bigcup S) \subseteq T$, in which case $U \neq \varnothing$, when taking $S = \varnothing$. Further, a subset of $\wp(A)$ is said to be *inductive*, whenever it is closed under unions of upward-directed subsets. Further, a *closure system over A* is any $\mathcal{C} \subseteq \wp(A)$ such that, for every $S \subseteq \mathcal{C}$, it holds that $(A \cap \bigcap S) \in \mathcal{C}$. In that case, any $\mathcal{B} \subseteq \mathcal{C}$ is called a *(closure) basis of* $\mathcal{C}$, provided $\mathcal{C} = \{A \cap \bigcap S | S \subseteq \mathcal{B}\}$. Furthermore, an *operator over A* is any unary operation O on $\wp(A)$. This is said to be *monotonic*, whenever it is $(\subseteq \cap \wp(A)^2)$-monotonic. Likewise, it is said to be *idempotent|transitive*, provided, for all $X \subseteq A$, it holds that $(X|O(O(X))) \subseteq O(X)$, respectively. Finally, it is said to be *inductive/finitary*, provided, for any upward-directed $U \subseteq \wp(A)$, it holds that $O(\bigcup U) \subseteq (\bigcup O[U])$. Then, a *closure operator over A* is any monotonic idempotent transitive operator over A, in which case $\mathrm{img}\, C$ is a[n inductive] closure system over A [iff C is inductive], determining C uniquely, as, for every basis $\mathcal{B}$ of $\mathrm{img}\, C$ (in particular, $\mathrm{img}\, C$ itself) and each $X \subseteq A$, $C(X) = (A \cap \bigcap\{Y \in \mathcal{B} | X \subseteq Y\})$, C and $\mathrm{img}\, C$ being said to be *dual* to one another.

2.2. **Algebraic background.** Unless otherwise specified, abstract algebras are denoted by Fraktur letters [possibly, with indices], their carriers (viz., underlying sets) being denoted by corresponding Italic letters [with same indices, if any].

A *(propositional/sentential) language|signature* is any algebraic (viz., functional) signature Σ (to be dealt with throughout the paper by default) constituted by function (viz., operation) symbols of finite arity to be treated as *(propositional/sentential) [primary] connectives*, the set of all nullary ones being denoted by $\Sigma{\restriction}0$.

Given a Σ-algebra $\mathfrak{A}$, the set $\mathrm{Con}(\mathfrak{A})$ of all *congruences of* $\mathfrak{A}$ (viz., equivalence relations θ on A such that *primary* operations of $\mathfrak{A}$ — i.e., those of the form $\varsigma^{\mathfrak{A}}$, where $\varsigma \in \Sigma$ — are θ-monotonic) is an inductive closure system over A^2, the dual closure operator (of congruence generation) being denoted by $\mathrm{Cg}^{\mathfrak{A}}$. Then, a *[partial] endomorphism of* $\mathfrak{A}$ is any homomorphism from [a subalgebra of] $\mathfrak{A}$ to $\mathfrak{A}$. Next, given any function f with $(\mathrm{dom}\, f) = A$ and $(\ker f) \in \mathrm{Con}(\mathfrak{A})$, we have the Σ-algebra $f[\mathfrak{A}]$ with carrier $f[A]$ and primary operations $\varsigma^{f[\mathfrak{A}]} \triangleq f[\varsigma^{\mathfrak{A}}]$, where $\varsigma \in \Sigma$. In particular, given any $\theta \in \mathrm{Con}(\mathfrak{A})$, $(\mathfrak{A}/\theta) \triangleq \nu_\theta[\mathfrak{A}]$ is known as the *quotient of* $\mathfrak{A}$ *by* θ. Finally, given a class K of Σ-algebras, set $\hom(\mathfrak{A}, \mathsf{K}) \triangleq (\bigcup\{\hom(\mathfrak{A}, \mathfrak{B}) \mid \mathfrak{B} \in \mathsf{K}\})$, in which case $\ker[\hom(\mathfrak{A}, \mathsf{K})] \subseteq \mathrm{Con}(\mathfrak{A})$, so $(A^2 \cap \bigcap \ker[\hom(\mathfrak{A}, \mathsf{K})]) \in \mathrm{Con}(\mathfrak{A})$.

Given any *rank*, viz., $\alpha \subseteq \omega$, put $\bar{x}_\alpha \triangleq \langle x_i \rangle_{i \in \alpha}$ and $\mathrm{Var}_\alpha \triangleq (\mathrm{img}\, \bar{x}_\alpha)$, elements of which being viewed as *(propositional/sentential) variables of rank* α. (In general, any mention of rank α within any context is normally omitted, whenever $\alpha = \omega$.)

Then, providing either $\alpha \neq \varnothing$ or Σ has a nullary connective, in which case α is called a Σ-*rank*, we have the absolutely-free Σ-algebra $\mathfrak{Fm}^{\alpha}_{\Sigma}$ freely-generated by the set Var_{α}, "its endomorphisms"/"elements of its carrier $\mathrm{Fm}^{\alpha}_{\Sigma}$ (viz., Σ-terms of rank α)" being called *(propositional|sentential)* Σ*-substitutions/-formulas of* $\{\Sigma\text{-}\}$*rank* α. In this way, *inverse* Σ*-substitutions of* $\{\Sigma\text{-}\}$*rank* α are functions of the form $\{\langle X, \sigma^{-1}[X]\rangle \mid X \subseteq \mathrm{Fm}^{\alpha}_{\Sigma}\}$, where σ is an endomorphism of $\mathfrak{Fm}^{\alpha}_{\Sigma}$. Any homomorphism h from $\mathfrak{Fm}^{\alpha}_{\Sigma}$ to a Σ-algebra $\mathfrak{A}(=\mathfrak{Fm}^{\alpha}_{\Sigma})$ is uniquely determined by {and so identified with} $h' = (h{\restriction}(\mathrm{Var}_{\alpha}(\setminus V)))$ (where $V \subseteq \mathrm{Var}_{\alpha}$ such that $h{\restriction}V$ is diagonal) as well as often written in the standard assignment (resp., substitution) form $[v/h(v)]_{v\in(\mathrm{dom}\, h')}$, $\varphi^{\mathfrak{A}}\langle[\rangle h\langle]\rangle$, where $\varphi \in \mathrm{Fm}^{\alpha}_{\Sigma}$, standing for $h(\varphi)$ (the algebra superscript being normally omitted just like in denoting primary operations of $\mathfrak{A}$). Then, given any $n \in \omega$, a *secondary n-ary connective of* Σ is any Σ-formula φ of Σ-rank $\rho_{\Sigma}(n) \triangleq (n + (1 - \min(1, \max(n, |\Sigma{\restriction}0|))))$, in which case, given any Σ-algebra $\mathfrak{A}$, an $f : A^n \to A$ is said to be *secondary/"(term-wise) definable* $\{$*by* $\varphi\}$*" of/in* $\mathfrak{A}$, provided, for all $\bar{a} \in A^{\rho_{\Sigma}(n)}$, it holds that $f(\bar{a}{\restriction}n) = \varphi^{\mathfrak{A}}[x_i/a_i]_{i\in\rho_{\Sigma}(n)}$. For the sake of formal unification, any primary n-ary connective $\varsigma \in \Sigma$ is identified with the secondary one $\varsigma(\bar{x}_n)$. A $\theta \in \mathrm{Con}(\mathfrak{Fm}^{\alpha}_{\Sigma})$ is said to be *fully-invariant*, if, for every Σ-substitution σ of rank α, it holds that $\sigma[\theta] \subseteq \theta$. Recall that, for any [surjective] $h \in \hom(\mathfrak{A}, \mathfrak{B})$, where $\mathfrak{A}$ and $\mathfrak{B}$ are Σ-algebras, it holds that:

$$\text{(2.1)} \qquad [\hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{B}) \subseteq]\{h \circ g \mid g \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A})\}) \subseteq \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{B}).$$

Any $\langle\phi, \psi\rangle \in \mathrm{Eq}^{\alpha}_{\Sigma} \triangleq (\mathrm{Fm}^{\alpha}_{\Sigma})^2$ is referred to as a Σ*-equation/-indentity of* $\{\Sigma\text{-}\}$*rank* α and normally written in the standard equational form $\phi \approx \psi$. In this way, given any $h \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A})$, $\ker h$ is the set of all Σ-identities of rank α *true/satisfied in* $\mathfrak{A}$ *under* h. Likewise, given a class K of Σ-algebras, $\theta^{\alpha}_{\mathsf{K}} \triangleq (\mathrm{Eq}^{\alpha}_{\Sigma} \cap \bigcap \ker[\hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathsf{K})]) \in \mathrm{Con}(\mathfrak{Fm}^{\alpha}_{\Sigma})$, being fully invariant, in view of (2.1), is the set of all all Σ-identities of rank α *true/satisfied in* K, in which case we set $\mathfrak{F}^{\alpha}_{\mathsf{K}} \triangleq (\mathfrak{Fm}^{\alpha}_{\Sigma}/\theta^{\alpha}_{\mathsf{K}})$. (In case α as well as both K and all elements of it are finite, the class $I = I^{\alpha}_{\mathsf{K}} \triangleq \{\langle\mathfrak{A}, h\rangle \mid \mathfrak{A} \in \mathsf{K}, h \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A})\}$ is a finite set — more precisely, $|I| = \sum_{\mathfrak{A}\in\mathsf{K}} |A|^{\alpha}$, in which case, putting, for each $i \in I$, $\mathfrak{A}_i \triangleq \pi_0(i) \in \mathsf{K}$, $h_i \triangleq \pi_1(i) \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A}_i)$ and $\mathfrak{B}_i \triangleq (\mathfrak{A}_i{\restriction}(\mathrm{img}\, h_i))$, we have $\hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \prod_{i\in I} \mathfrak{B}_i) \ni g : \mathrm{Fm}^{\alpha}_{\Sigma} \to (\prod_{i\in I} B_i), \varphi \mapsto \langle h_i(\varphi)\rangle_{i\in I}$ with $(\ker g) = \theta \triangleq \theta^{\alpha}_{\mathsf{K}}$, and so, by the Homomorphism Theorem, $e \triangleq (\nu^{-1}_{\theta} \circ g)$ is an isomorphism from $\mathfrak{F}^{\alpha}_{\mathsf{K}}$ onto the subdirect product $(\prod_{i\in I} \mathfrak{B}_i){\restriction}(\mathrm{img}\, g)$ of $\langle\mathfrak{B}_i\rangle_{i\in I}$. In this way, the former is finite, for the latter is so — more precisely, $|F^{\alpha}_{\mathsf{K}}| \leqslant (\max\{|A| \mid \mathfrak{A} \in \mathsf{K}\}^{|I|}$.)

The class of all Σ-algebras satisfying every element of an $\mathcal{E} \subseteq \mathrm{Eq}^{\omega}_{\Sigma}$ is called the *variety axiomatized by* $\mathcal{E}$. Then, the variety $\mathbf{V}(\mathsf{K})$ axiomatized by $\theta^{\omega}_{\mathsf{K}}$ is the least variety including K and is said to be *generated by* K, in which case $\theta^{\alpha}_{\mathbf{V}(\mathsf{K})} = \theta^{\alpha}_{\mathsf{K}}$, and so $\mathfrak{F}^{\alpha}_{\mathbf{V}(\mathsf{K})} = \mathfrak{F}^{\alpha}_{\mathsf{K}}$.

Given a fully invariant $\theta \in \mathrm{Con}(\mathfrak{Fm}^{\omega}_{\Sigma})$, by (2.1), $\mathfrak{Fm}^{\omega}_{\Sigma}/\theta$ belongs to the variety V axiomatized by θ, in which case any Σ-identity satisfied in V belongs to θ, and so $\theta^{\omega}_{\mathsf{V}} = \theta$. In particular, given a variety V of Σ-algebras, we have $\mathfrak{F}^{\alpha}_{\mathsf{V}} \in \mathsf{V}$.

Finally, let $\mathrm{Var} : \mathrm{Fm}^{\omega}_{\Sigma} \to \wp_{\omega}(\mathrm{Var}_{\omega})$ be the mapping assigning the set of all *actually* occurring variables.

2.2.1. *Lattice-theoretic background.*

2.2.1.1. Semi-lattices. Let $\diamond$ be a (possibly, secondary) binary connective of Σ.

A Σ-algebra $\mathfrak{A}$ is called a $\diamond$*-semi-lattice*, provided it satisfies *semi-lattice* identities *for* $\diamond$ (viz., *idempotence* $(x_0 \diamond x_0) \approx x_0$, *commutativity* $(x_0 \diamond x_1) \approx (x_1 \diamond x_0)$ and *associativity* $(x_0 \diamond (x_1 \diamond x_2)) \approx ((x_0 \diamond x_1) \diamond x_2)$ ones), in which case we have the partial ordering $\leq^{\mathfrak{A}}_{\diamond}$ on A, given by $(a \leq^{\mathfrak{A}}_{\diamond} b) \overset{\text{def}}{\iff} (a = (a \diamond^{\mathfrak{A}} b))$, for all $a, b \in A$. Then, in case the [dual] poset $\langle A, (\leq^{\mathfrak{A}}_{\diamond})^{[-1]}\rangle$ has the least element (viz., lower *bound*), this is called the *[dual] $\langle\diamond-\rangle$bound of* $\mathfrak{A}$ and denoted by $[\delta]\beta^{\mathfrak{A}}_{\diamond}$, while $\mathfrak{A}$ is referred to as a *$\diamond$-semi-lattice with [dual] bound* $\{a$, whenever $a = [\delta]\beta^{\mathfrak{A}}_{\diamond}\}$.

Lemma 2.1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be $\diamond$-semi-lattices with bound and $h \in \hom(\mathfrak{A}, \mathfrak{B})$. Suppose $h[A] = B$. Then, $h(\beta^{\mathfrak{A}}_{\diamond}) = \beta^{\mathfrak{B}}_{\diamond}$.*

Proof. There is some $a \in A$ such that $h(a) = \beta^{\mathfrak{B}}_{\diamond}$, in which case $(a \diamond^{\mathfrak{A}} \beta^{\mathfrak{A}}_{\diamond}) = \beta^{\mathfrak{A}}_{\diamond}$, so $h(\beta^{\mathfrak{A}}_{\diamond}) = (h(a) \diamond^{\mathfrak{B}} h(\beta^{\mathfrak{A}}_{\diamond})) = (\beta^{\mathfrak{B}}_{\diamond} \diamond^{\mathfrak{B}} h(\beta^{\mathfrak{A}}_{\diamond})) = \beta^{\mathfrak{B}}_{\diamond}$, as required. □

2.2.1.2. Lattices. Let $\barwedge$ and $\veebar$ be (possibly, secondary) binary connectives of Σ.

A Σ-algebra $\mathfrak{A}$ is called a *[distributive] $(\barwedge, \veebar)$-lattice*, provided it satisfies *[distributive] lattice* identities *for $\barwedge$ and $\veebar$* (viz., semi-lattice identities for both $\barwedge$ and $\veebar$ as well as *absorption* $(x_0 \diamond_0 (x_0 \diamond_1 x_1)) \approx x_0$ [and *distributivity* $(x_0 \diamond_0 (x_1 \diamond_1 x_2) \approx ((x_0 \diamond_0 x_1) \diamond_1 (x_0 \diamond_0 x_2))$] identities *for $\barwedge$ and $\veebar$*, for all bijective $\bar{\diamond} : 2 \to \{\barwedge, \veebar\}$), in which case $\leq^{\mathfrak{A}}_{\barwedge}$ and $\leq^{\mathfrak{A}}_{\veebar}$ are inverse/dual to one another, and so, in case $\mathfrak{A}$ is a $\veebar$-semi-lattice with bound (in particular, when A is finite), $\beta^{\mathfrak{A}}_{\veebar}$ is the dual $\barwedge$-bound of $\mathfrak{A}$ (viz., the greatest element of the poset $\langle A, \leq^{\mathfrak{A}}_{\barwedge}\rangle$). Then, in case $\mathfrak{A}$ is a $\{$distributive$\}$ $(\barwedge, \veebar)$-lattice, it is said to be that *with zero$|$unit (a)*, whenever it is a $(\barwedge|\veebar)$-semi-lattice with bound (a).

2.2.1.2.1. Bounded lattices. Let $\Sigma_{\langle\emptyset\rangle\{+\}[01]} \triangleq (\varnothing\{\cup\{\wedge, \vee\}\}[\cup\{\bot, \top\}])$ be the $\{$[bounded] lattice$\}$ signature $\{$with binary $\wedge$ (conjunction) and $\vee$ (disjunction)$\}$ [$\{$as well as$\}$ with nullary $\bot$ and $\top$ (falsehood/zero and truth/unit constants, respectively)]. Then, a $\Sigma_{+[01]}$-algebra $\mathfrak{A}$ is called a *[bounded] (distributive) lattice*, whenever it is a (distributive) $(\wedge, \vee)$-lattice [with zero $\bot^{\mathfrak{A}}$ and unit $\top^{\mathfrak{A}}$] $\{$cf., e.g., [1]$\}$. Given any signature $\Sigma' \supseteq \Sigma_+$ and any $\phi, \psi \in \mathrm{Fm}^{\omega}_{\Sigma'}$, $\phi \lessapprox \psi$ stands for $\phi \approx (\phi \wedge \psi)$. Likewise, given any Σ'-algebra $\mathfrak{A}$ with Σ_+-reduct being a lattice, $\leq^{\mathfrak{A}}$ stands for $\leq^{\mathfrak{A}}_{\wedge}$. Then, given any $n \in (\omega \setminus 2)$, $\mathfrak{D}_{n[01]}$ denotes the [bounded] distributive lattice with carrier $(n \div (n-1)) \triangleq \{\frac{m}{n-1} \mid m \in n\}$ and $\leq^{\mathfrak{D}_{n[01]}} \triangleq (\leqslant \cap D^2_{n[01]})$.

2.3. **Logical background.**

2.3.1. *Propositional calculi and logics.* A *(propositional$\|$sentential) [finitary$|$unary$|$axiomatic] Σ-rule/-calculus* $\{$*of $\langle\Sigma-\rangle$-rank α*$\}$ is any element/subset of the set $\wp_{[\omega|(2\setminus 1)|1]}(\mathrm{Fm}^{\omega\{\cap\alpha\}}_{\Sigma}) \times \mathrm{Fm}^{\omega\{\cap\alpha\}}_{\Sigma}$, any Σ-rule $\langle\Gamma, \varphi\rangle$ being normally written in the standard sequent form $\Gamma \vdash \varphi$, "the left"/"any element of the right" component$|$side of it being referred to as the/a *conclusion/premise of* it. Then, we set $\sigma(\Gamma \vdash \varphi) \triangleq (\sigma[\Gamma] \vdash \sigma(\varphi))$, where σ is a Σ-substitution. Axiomatic Σ-rules are called *(propositional/sentential) Σ-axioms* and are identified with their conclusions.

A *(propositional/sentential) Σ-logic* (cf., e.g., [7]) is any closure operator C over $\mathrm{Fm}^{\omega}_{\Sigma}$ that is *structural* in the sense that $\sigma[C(X)] \subseteq C(\sigma[X])$, for all $X \subseteq \mathrm{Fm}^{\omega}_{\Sigma}$ and all $\sigma \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{Fm}^{\omega}_{\Sigma})$, that is, $\mathrm{img}\, C$ is closed under inverse Σ-substitutions. Then, we have the equivalence relation $\equiv^{\alpha}_{C} \triangleq \{\langle\phi, \psi\rangle \in \mathrm{Eq}^{\alpha}_{\Sigma} \mid C(\phi) = C(\psi)\}$ on $\mathrm{Fm}^{\alpha}_{\Sigma}$, where α is a Σ-rank, called the *inter-derivablity relation of C*, whenever $\alpha = \omega$. A *congruence of C* is any $\theta \in \mathrm{Con}(\mathfrak{Fm}^{\omega}_{\Sigma})$ such that $\theta \subseteq \equiv^{\omega}_{C}$, the set of all them being denoted by $\mathrm{Con}(C)$. Then, given any $\theta, \vartheta \in \mathrm{Con}(C)$, $\mathrm{Tr}(\theta \cup \vartheta)$, being well-known to be a congruence of $\mathfrak{Fm}^{\omega}_{\Sigma}$, is then that of C, for θ^{ω}_{C}, being an equivalence relation, is transitive. In particular, any maximal congruence of C (that exists, by Zorn Lemma, because $\mathrm{Con}(C) \ni \Delta_{\mathrm{Fm}^{\omega}_{\Sigma}}$ is both non-empty and inductive, for $\mathrm{Con}(\mathfrak{Fm}^{\omega}_{\Sigma})$ is so) is the greatest one to be denoted by $\partial(C)$. Then, C is said to be *self-extensional*, whenever $\equiv^{\omega}_{C} \in \mathrm{Con}(\mathfrak{Fm}^{\omega}_{\Sigma})$. that is, $\partial(C) = \equiv^{\omega}_{C}$.

Definition 2.2 (cf. [16])**.** Given a Σ-logic C, the variety $\mathrm{IV}(C)$ axiomatized by $\partial(C)$ is called the *intrinsic variety of C*. □

Next, a Σ-rule $\Gamma \to \Phi$ is said to be *satisfied/derivable in* a Σ-logic C, provided $\Phi \in C(\Gamma)$, Σ-axioms satisfied in C being referred to as *theorems of* C.

Definition 2.3. A Σ-logic C' is said to be a *(proper) [K-]extension of* a Σ-logic C [where $K \subseteq \infty$], whenever ($C' \neq C$ and) $C(X) \subseteq C'(X)$, for all $X \in \wp_{[K]}(\mathrm{Fm}^{\omega}_{\Sigma})$, C being said to be a *(proper) [K-]sublogic of* C', in which case C' is said to be *axiomatized by* a Σ-calculus $\mathcal{C}$ *relatively to* C, whenever C' is the least (w.r.t. the extension partial ordering) extension of C satisfying every rule in $\mathcal{C}$, while C' is said to be *({C-relatively} maximally)/ [inferentially] consistent/inconsistent*, provided $x_1 \notin / \in C(\varnothing[\cup\{x_0\}])$, (and every [inferentially] consistent extension of $C'\{\cap C\}$ is a sublogic of C')/"in which case $\equiv^{\omega}_{C} = \mathrm{Eq}^{\omega}_{\Sigma} \in \mathrm{Con}(\mathfrak{Fm}^{\omega}_{\Sigma})$, and so C is self-extensional", the only inconsistent Σ-logic being denoted by IC^{Σ}. Likewise, C' and C are said to be *axiomatically-equivalent*, whenever $C'(\varnothing) = C(\varnothing)$. □

Further, a Σ-rule $\mathcal{R}$ is said to be *admissible in* a Σ-logic C, provided the extension of C relatively axiomatized by $\mathcal{R}$ is axiomatically-equivalent to C. Clearly, $\mathcal{R}$ is admissible in C, whenever it is derivable in C. If the converse holds *in general*, that is, every Σ-rule is derivable in C, whenever it is admissible in C, then C is said to be *structurally/deductively/inferentially complete|maximal*. Clearly, C is structurally complete iff it has no proper axiomatically-equivalent extension. In general, $(\bigcap_{C' \in S}(\mathrm{img}\, C')) \ni C(\varnothing)$, where $S \ni C$ is the set of all Σ-logics axiomatically-equivalent to C, is a closure system over $\mathrm{Fm}^{\omega}_{\Sigma}$ closed under inverse Σ-substitutions, in which case the dual closure operator over $\mathrm{Fm}^{\omega}_{\Sigma}$ is the greatest axiomatically-equivalent (and so structurally complete) extension of C, called the *structural completion of* C.

Next, C is said to be *(strongly)/weakly {classically} $\bar{\wedge}$-conjunctive*, provided $C(\{x_0, x_1\}) = / \subseteq C(x_0 \bar{\wedge} x_1)$. Likewise, C is said to be *(strongly)/weakly {classically} $\veebar$-disjunctive*, if $C(X \cup \{\phi \veebar \psi\}) = / \subseteq (C(X \cup \{\phi\}) \cap C(X \cup \{\psi\}))$, where $(X \cup \{\phi, \psi\}) \subseteq \mathrm{Fm}^{\omega}_{\Sigma}$, "in which case"/"that is, the first two — viz., (2.2) —- of" the following four rules:

$$x_i \vdash (x_0 \veebar x_1), \quad \text{where } i \in 2, \tag{2.2}$$

$$(x_0 \veebar x_1) \vdash (x_1 \veebar x_0), \tag{2.3}$$

$$(x_0 \veebar x_0) \vdash x_0 \tag{2.4}$$

are satisfied in C. Further, C is said to *have/satisfy Deduction Theorem (DT) with respect to* a (possibly, secondary) binary connective $\sqsupset$ of Σ (fixed throughout the paper by default), provided, for all $\phi \in X \subseteq \mathrm{Fm}^{\omega}_{\Sigma}$ and all $\psi \in C(X)$, it holds that $(\phi \sqsupset \psi) \in C(X \setminus \{\phi\})$, in which case the following axioms:

$$x_0 \sqsupset x_0, \tag{2.5}$$

$$x_0 \sqsupset (x_1 \sqsupset x_0) \tag{2.6}$$

are satisfied in C. Then, C is said to be *weakly {classically} $\sqsupset$-implicative*, if it has DT w.r.t. $\sqsupset$ as well as satisfies the *Modus Ponens* rule:

$$\{x_0, x_0 \sqsupset x_1\} \vdash x_1, \tag{2.7}$$

in which case the following axiom:

$$(x_0 \uplus_{\sqsupset} (x_0 \sqsupset x_1)), \tag{2.8}$$

where $(x_0 \uplus_{\sqsupset} x_1) \triangleq ((x_0 \sqsupset x_1) \sqsupset x_1)$ is the *intrinsic disjunction of (implication)* $\sqsupset$, is satisfied in C. Likewise, C is said to be *(strongly) {classically} $\sqsupset$-implicative*, whenever it is weakly so and satisfies the *Peirce Law* axiom (cf. [12]):

$$((x_0 \sqsupset x_1) \uplus_{\sqsupset} x_0). \tag{2.9}$$

Furthermore, C is said to be *[maximally] $\wr$-paraconsistent* [cf. [15] as well as the reference [Pyn95 b] therein], where $\wr$ is a (possibly, secondary) unary connective of Σ, tacitly fixed throughout the paper by default, provided it does not satisfy the *Ex Contradictione Quodlibet* rule:

$$\{x_0, \wr x_0\} \vdash x_1 \tag{2.10}$$

[and has no proper $\wr$-paraconsistent extension]. Likewise, C is said to be {*maximally*} *[inferentially]* $(\veebar, \wr)$*-paracomplete*, whenever it does not satisfy the [inferential version of] the *Excluded Middle Law* axiom

$$[x_1 \vdash](x_0 \veebar \wr x_0) \tag{2.11}$$

{and has no proper [inferentially] $(\veebar, \wr)$-paracomplete extension}. Given any $\Sigma' \subseteq \Sigma$, the Σ'-logic C', defined by $C'(X) \triangleq (\mathrm{Fm}^{\omega}_{\Sigma'} \cap C(X))$, for all $X \subseteq \mathrm{Fm}^{\omega}_{\Sigma'}$, is called the *$\Sigma'$-fragment of C*, C being referred to as a *(Σ-)expansion of C'*, in which case $\equiv^{\omega}_{C'} = (\equiv^{\omega}_{C} \cap \mathrm{Eq}^{\omega}_{\Sigma'})$, and so C' is self-extensional, whenever C is so. Finally, C is said to be *theorem-less/purely-inferential*, whenever it has no theorem, that is, $\varnothing \in (\mathrm{img}\, C)$. In general, $(\mathrm{img}\, C) \cup \{\varnothing\}$ is a closure system over $\mathrm{Fm}^{\omega}_{\Sigma}$ closed under inverse Σ-substitutions, for $\mathrm{img}\, C$ is so, in which case the dual closure operator C_{+0} over $\mathrm{Fm}^{\omega}_{\Sigma}$ is the greatest purely-inferential sublogic of C, called the *purely-inferential version of C* and being an $(\infty \setminus 1)$-extension of C (cf. Definition 2.3), so

$$\equiv^{\omega}_{C} = \equiv^{\omega}_{C_{+0}} \tag{2.12}$$

(in particular, C_{+0} is self-extensional iff C is so).

Remark 2.4. Let C be a Σ-logic and $\phi \in C(\varnothing)$, in which case, by the structurality of C, $\psi \triangleq (\phi[x_i/x_0]_{i\in\omega}) \in (\mathrm{Fm}^{1}_{\Sigma} \cap C(\varnothing))$, and so C is weakly ψ-disjunctive. □

2.3.2. *Logical matrices.* A *(logical) Σ-matrix* (cf., e.g., [7]) is any pair of the form $\mathcal{A} = \langle \mathfrak{A}, D^{\mathcal{A}} \rangle$, where $\mathfrak{A}$ is a Σ-algebra, called the *underlying algebra of $\mathcal{A}$*, while A is called the *carrier/"underlying set" of $\mathcal{A}$*, whereas $D^{\mathcal{A}} \subseteq A$ is called the *truth predicate of $\mathcal{A}$*, elements of $A[\cap D^{\mathcal{A}}]$ being referred to as *[distinguished] values of $\mathcal{A}$*. (In general, matrices are denoted by Calligraphic letters [possibly, with indices], their underlying algebras being denoted by corresponding capital Fraktur letters [with same indices, if any].) This is said to be *[no-more/less-than-]n-valued*, where $n \in (\omega \setminus 1)$, provided $|A| = [\leqslant/\geqslant]n$. Next, it is said to be *[in]consistent*, whenever $D^{\mathcal{A}} \neq [=]A$, respectively. Likewise, it is is said to be *truth[-non]-empty*, whenever $D^{\mathcal{A}} = [\neq]\varnothing$. Further, it is said to be *truth-/false-singular*, if $|((D^{\mathcal{A}}/(A \setminus D^{\mathcal{A}}))| \in 2$. Finally, $\mathcal{A}$ is said to be *finite[ly generated]/"generated by $B \subseteq A$"*, if $\mathfrak{A}$ is so.

Given any Σ-rank α and any class M of Σ-matrices, we have the closure operator $\mathrm{Cn}^{\alpha}_{\mathsf{M}}$ over $\mathrm{Fm}^{\alpha}_{\Sigma}$ dual to the closure system with basis $\mathcal{B}^{\alpha}_{\mathsf{M}} \triangleq \{h^{-1}[D^{\mathcal{A}}] \mid \mathcal{A} \in \mathsf{M}, h \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A})\}$, in which case:

$$\mathrm{Cn}^{\alpha}_{\mathsf{M}}(X) = (\mathrm{Fm}^{\alpha}_{\Sigma} \cap \mathrm{Cn}^{\omega}_{\mathsf{M}}(X)), \tag{2.13}$$

for all $X \subseteq \mathrm{Fm}^{\alpha}_{\Sigma}$. Then, by (2.1), $\mathrm{Cn}^{\omega}_{\mathsf{M}}$ is a Σ-logic, called the *logic of/"defined by"* M. A Σ-logic is said to be {*"unitary‖uniform[ly]"|double|finitely*} *(no-more/less-than-)n-valued*, where $n \in (\omega \setminus 1)$, whenever it is defined by a {one-element|two-element|finite} class of (no-more/less-than-)n-valued Σ-matrices /{in which case it is finitary, as the logic of any finite set of finite Σ-matrices is so; cf. [7]}. Then, a [uniform{ly}] n-valued Σ-logic, where $n \in (\omega \setminus 2)$, is said to be *minimal(ly)* so, unless it is [uniformly] no-more-than-$(n-1)$-valued.

As usual, Σ-matrices are treated as first-order model structures (viz., algebraic systems; cf. [10]) of the first-order signature $\Sigma \cup \{D\}$ with unary predicate D, in which case any [in]finitary Σ-rule $\Gamma \vdash \phi$ is viewed as the [in]finitary equality-free basic strict Horn formula $(\bigwedge \Gamma) \to \phi$ under the standard identification of any propositional Σ-formula ψ with the first-order atomic formula $D(\psi)$, as well as is *true/satisfied in* a class M of Σ-matrices (in the conventional model-theoretic sense; cf., e.g., [10]) iff it is satisfied in the logic of M, theorems of which being referred to as *tautologies of* M.

Remark 2.5. Since any rule with[out] premises is [not] true in any truth-empty matrix, given any class M of Σ-matrices, the theorem-less version of the logic of M is defined by that of the form by $\mathsf{M}\cup\mathsf{S}$ with only truth-empty elements of $\mathsf{S} \neq \varnothing$. □

Let $\mathcal{A}$ and $\mathcal{B}$ be two Σ-matrices. A *(strict) [surjective] {injective} homomorphism from $\mathcal{A}$ [on]to $\mathcal{B}$* is any {injective} $h \in \hom(\mathfrak{A},\mathfrak{B})$ such that $[h[A] = B$ and] $D^{\mathcal{A}} \subseteq h^{-1}[D^{\mathcal{B}}](\subseteq D^{\mathcal{A}})$, the set of all them being denoted by $\hom^{[\mathrm{S}]}_{(\mathrm{S})}(\mathcal{A},\mathcal{B})$, in which case $\mathcal{B}/\mathcal{A}$ is said to be a *(strictly) [surjectively] {injectively} homomorphic image/counter-image* ([{as well as called an *isomorphic copy*}]) *of* $\mathcal{A}/\mathcal{B}$, respectively. Then, by (2.1), we have:

$$(\hom^{[\mathrm{S}]}_{\mathrm{S}}(\mathcal{A},\mathcal{B}) \neq \varnothing) \quad\Rightarrow\quad (\mathrm{Cn}^{\alpha}_{\mathcal{B}}(X) \subseteq \mathrm{Cn}^{\alpha}_{\mathcal{A}}(X)[\subseteq \mathrm{Cn}^{\alpha}_{\mathcal{B}}(X)]), \tag{2.14}$$

$$(\hom^{\mathrm{S}}(\mathcal{A},\mathcal{B}) \neq \varnothing) \quad\Rightarrow\quad (\mathrm{Cn}^{\alpha}_{\mathcal{A}}(\varnothing) \subseteq \mathrm{Cn}^{\alpha}_{\mathcal{B}}(\varnothing)), \tag{2.15}$$

for all Σ-ranks α and all $X \subseteq \mathrm{Fm}^{\alpha}_{\Sigma}$. Further, $\mathcal{A}[\neq \mathcal{B}]$ is said to be a *[proper] submatrix of* $\mathcal{B}$, whenever $\Delta_A \in \hom_{\mathrm{S}}(\mathcal{A},\mathcal{B})$, in which case we set $(\mathcal{B}{\restriction}A) \triangleq \mathcal{A}$. Injective/bijective strict homomorphisms from $\mathcal{A}$ to $\mathcal{B}$ are called *embeddings/isomorphisms of/from $\mathcal{A}$ into/onto $\mathcal{B}$*, in case of existence of which $\mathcal{A}$ is said to be *embeddable/isomorphic into/to $\mathcal{B}$*.

Given a Σ-matrix $\mathcal{A}$, $(\chi^{\mathcal{A}}/\theta^{\mathcal{A}}) \triangleq (\chi^{D^{\mathcal{A}}}_{A})/(\ker\chi^{\mathcal{A}}))$ is referred to as the *characteristic function/relation of* $\mathcal{A}$. Then, any $\theta \in \mathrm{Con}(\mathfrak{A})$ such that $\theta \subseteq \theta^{\mathcal{A}}$, in which case ν_θ is a strict surjective homomorphism from $\mathcal{A}$ onto $(\mathcal{A}/\theta) \triangleq \langle\mathfrak{A}/\theta, D^{\mathcal{A}}/\theta\rangle$, is called a *congruence of* $\mathcal{A}$, the set of all them being denoted by $\mathrm{Con}(\mathcal{A})$. Given any $\theta,\vartheta \in \mathrm{Con}(\mathcal{A})$, $\mathrm{Tr}(\theta\cup\vartheta)$, being well-known to be a congruence of $\mathfrak{A}$, is then that of $\mathcal{A}$, for $\theta^{\mathcal{A}}$, being an equivalence relation, is transitive. In particular, any maximal congruence of $\mathcal{A}$ (that exists, by Zorn Lemma, because $\mathrm{Con}(\mathcal{A}) \ni \Delta_A$ is both non-empty and inductive, for $\mathrm{Con}(\mathfrak{A})$ is so) is the greatest one to be denoted by $\partial(\mathcal{A})$, that is traditionally called the *Leibniz* congruence of $\mathcal{A}$ but denoted, for quite unclear reasons, by rather $\Omega(\mathcal{A})$ than, e.g., $\Lambda(\mathcal{A})$ (here we though naturally adapt more coherent conventions adopted in [23] to use its results *immediately*). Finally, $\mathcal{A}$ is said to be *[(finitely) hereditarily] simple*, whenever it has no non-diagonal congruence [as well as no non-simple (finitely-generated) submatrix].

Remark 2.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be two Σ-matrices and $h \in \hom(\mathcal{A},\mathcal{B})$ strict [and surjective]. Then,* $\chi^{\mathcal{A}} = (h\circ\chi^{\mathcal{B}})$ (in particular, $\theta^{\mathcal{A}} = h^{-1}[\theta^{\mathcal{B}}]$) and, for every $\theta \in \mathrm{Con}(\mathfrak{B})$, $h^{-1}[\theta] \in \mathrm{Con}(\mathfrak{A})$ [while $h[h^{-1}[\theta]] = \theta$]. Therefore:

(i) *for every $\theta \in \mathrm{Con}(\mathcal{B})$, $h^{-1}[\theta] \in \mathrm{Con}(\mathcal{A})$ [while $h[h^{-1}[\theta]] = \theta$].*

In particular (when $\theta = \Delta_B$), by (i), we have $(\ker h) = h^{-1}[\Delta_B] \in \mathrm{Con}(\mathcal{A})$, so:

(ii) *h is injective, whenever $\mathcal{A}$ is simple.*

[Likewise, for any $\theta \in \mathrm{Con}(\mathcal{B})$, by (i), we have $h^{-1}[\theta] \in \mathrm{Con}(\mathcal{A})$, in which case we get $h^{-1}[\theta] \subseteq \partial(\mathcal{A})$, and so, by (i), we eventually get $\theta = h[h^{-1}[\theta]] \subseteq h[\partial(\mathcal{A})]$ (in particular, $\Delta_B \subseteq \theta \subseteq \Delta_B$, whenever $\partial(\mathcal{A}) \subseteq (\ker h)$).] Thus:

[(iii) *$\mathcal{B}$ is simple, whenever $\mathcal{A}$ is so.*]

(iv) *$\mathcal{A}/\partial(\mathcal{A})$ is simple.* □

Definition 2.7. A Σ-matrix $\mathcal{A}$ is said to be a *[K-]model of* a Σ-logic C {*over* $\mathfrak{A}$} [where $K \subseteq \infty$], provided C is a [K-]sublogic the logic of $\mathcal{A}$ ⟨cf. Definition 2.3⟩, the class of all (simple of) them being denoted by $\mathrm{Mod}^{(*)}_{[K]}(C\{,\mathfrak{A}\})$, respectively. Then,

$\mathrm{Fi}_C(\mathfrak{A}) \triangleq \pi_1[\mathrm{Mod}(C,\mathfrak{A})]$, whose elements are called *filters of C over $\mathfrak{A}$*, is a closure system over A, $\mathrm{Fg}^{\mathfrak{A}}_C$ denoting the dual closure operator ⟨of filter generation⟩. □

A Σ-matrix $\mathcal{A}$ is said to be *$\wr$-paraconsistent/"[inferentially] $(\veebar,\wr)$-paracomplete"*, whenever its logic is so. Next, $\mathcal{A}$ is said to be *(strongly)/weakly {classically} $\diamond$-conjunctive*, provided $(\{a,b\} \subseteq D^{\mathcal{A}}) \Leftrightarrow / \Leftarrow ((a \diamond^{\mathfrak{A}} b) \in D^{\mathcal{A}})$, for all $a,b \in A$, that is, the logic of $\mathcal{A}$ is strongly/weakly $\diamond$-conjunctive. Then, $\mathcal{A}$ is said to be *(strongly)/weakly {classically} $\diamond$-disjunctive*, whenever $\langle \mathfrak{A}, A \setminus D^{\mathcal{A}}\rangle$ is strongly/weakly $\diamond$-conjunctive, "in which case"/"that is," the logic of $\mathcal{A}$ is strongly/weakly $\diamond$-disjunctive, and so is the logic of any class of strongly/weakly $\diamond$-disjunctive Σ-matrices. Likewise, $\mathcal{A}$ is said to be *(weakly/strongly){classically} $\sqsupset$-implicative*, whenever $((a \in D^{\mathcal{A}}) \Rightarrow (b \in D^{\mathcal{A}})) \Leftrightarrow ((a \sqsupset^{\mathfrak{A}} b) \in D^{\mathcal{A}})$, for all $a,b \in A$, in which case it is $\uplus_{\sqsupset}$-disjunctive, while the logic of $\mathcal{A}$ is $\sqsupset$-implicative, for both (2.7) and (2.9) are true in any $\sqsupset$-implicative (and so $\uplus_{\sqsupset}$-disjunctive) Σ-matrix, while DT is immediate, and so is the logic of any class of $\sqsupset$-implicative Σ-matrices. Furthermore, given any $\Sigma' \subseteq \Sigma$, $\mathcal{A}$ is said to be a *(Σ-)expansion of* its *Σ'-reduct* $(\mathcal{A}{\restriction}\Sigma') \triangleq \langle \mathfrak{A}{\restriction}\Sigma', D^{\mathcal{A}}\rangle$, clearly defining the Σ'-fragment of the logic of $\mathcal{A}$. Finally, $\mathcal{A}$ is said to be *weakly/(strongly) {classically} $\wr$-negative*, provided, for all $a \in A$, $(a \in D^{\mathcal{A}}) \Leftarrow / \Leftrightarrow (\wr^{\mathfrak{A}} a \notin D^{\mathcal{A}})$, in which case it is truth-non-empty/", and so consistent".

Remark 2.8. For any Σ-matrices $\mathcal{A}$ and $\mathcal{B}$, the following hold:

(i) $\mathcal{A}$ is:
  (a) [weakly] $\diamond$-disjunctive/-conjunctive iff it is [weakly] $\diamond^{\wr}$-conjunctive/-disjunctive, respectively, whenever it is $\wr$-negative, where $(x_0 \diamond^{\wr} x_1) \triangleq \wr(\wr x_0 \diamond \wr x_1)$ is the *$\wr$-dual|De-Morgan counterpart of* $\diamond$;
  (b) $\sqsupset^{\wr}_{\diamond}$-implicative, if it is both $\wr$-negative and $\diamond$-disjunctive, where $(x_0 \sqsupset^{\wr}_{\diamond} x_1) \triangleq (\wr x_0 \diamond x_1)$ is the *material implication of/"defined|given by" {negation} $\wr$ and {disjunction} $\diamond$*.
  (c) not $\wr$-paraconsistent, whenever it is $\wr$-negative;
  (d) not $(\diamond,\wr)$-paracomplete, whenever it is both weakly $\wr$-negative and weakly $\diamond$-disjunctive;

(ii) for any strict [surjective] (injective) $h \in \hom(\mathcal{A},\mathcal{B})$, the following hold:
  (a) $\mathcal{A}$ is {weakly} $\wr$-negative|$\diamond$-conjunctive/-disjunctive/-implicative if[f] $\mathcal{B}$ is so;
  (b) $\mathcal{B}$ is consistent/truth-non-empty if[f] $\mathcal{A}$ is so;
  (c) $\mathcal{A}$ is false-/truth-singular (if [and]) [only if] $\mathcal{B}$ is so. □

Given a set I and an I-tuple $\overline{\mathcal{A}}$ of Σ-matrices, [any submatrix $\mathcal{B}$ of] the Σ-matrix $(\prod_{i\in I} \mathcal{A}_i) \triangleq \langle \prod_{i\in I} \mathfrak{A}_i, \prod_{i\in I} D^{\mathcal{A}_i}\rangle$ is called the [a] *[sub]direct product of* $\overline{\mathcal{A}}$ [whenever, for each $i \in I$, $\pi_i[B] = A_i$]. As usual, if $(\mathrm{img}\,\overline{\mathcal{A}}) \subseteq \{\mathcal{A}\}$, where $\mathcal{A}$ is a Σ-matrix, we set $\mathcal{A}^I \triangleq (\prod_{i\in I} \mathcal{A}_i)$.

Given a class M of Σ-matrices, the class of all "strictly surjectively homomorphic [counter-]images"/ "isomorphic copies"/"(consistent) submatrices" of elements of M is denoted by $(\mathbf{H}^{[-1]}/\mathbf{I}/\mathbf{S}_{(*)})(\mathsf{M})$, respectively. Likewise, the class of all [sub]direct products of tuples (of cardinality $\in K \subseteq \infty$) constituted by elements of M is denoted by $\mathbf{P}^{[\mathrm{SD}]}_{(K)}(\mathsf{M})$.

2.3.2.1. Classical matrices and logics. Σ-matrices with diagonal characteristic function (and so relation) are said to be *classically-canonical*, isomorphisms between them being diagonal, in which case isomorphic ones being equal. Then, the characteristic function of any Σ-matrix $\mathcal{A}$ with diagonal characteristic relation — viz., injective characteristic function — (and so no-more-than-two-valued) is an isomorphism from it onto the classically-canonical Σ-matrix $\mathbb{C}(\mathcal{A}) \triangleq \langle \chi^{\mathcal{A}}[\mathfrak{A}], \{1\} \rangle$, called the *[classical] canonization of* $\mathcal{A}$.

A (classically-canonical) two-valued Σ-matrix $\mathcal{A}$ [with functionally complete underlying algebra] is said to be *[genuinely] (canonical{ly}) $\wr$-classical*, whenever it is $\wr$-negative, in which case it is both false- and truth-singular (and so its characteristic relation is diagonal) but is not $\wr$-paraconsistent, by Remark 2.8(i)(c).

A Σ-logic is said to be *(genuinely) $\wr$-[sub]classical*, whenever it is [a sublogic of] the logic of a (genuinely) $\wr$-classical Σ-matrix, in which case it is inferentially consistent. Then, a Σ-matrix is said to be *$\wr$-classically-defining*, whenever its logic is $\wr$-classical. Likewise, a unary $\sim \in \Sigma$ is called a *subclassical negation for* a Σ-logic C, whenever the $\sim$-fragment of C is $\sim$-subclassical, in which case:

$$\sim^m x_0 \notin C(\sim^n x_0), \tag{2.16}$$

for all $m, n \in \omega$ such that the integer $m - n$ is odd, where the secondary unary connective $\wr^l$ of Σ is defined by induction on $l \in \omega$ via setting $\wr^{0[+l+1]} x_0 \triangleq [\wr \wr^l] x_0$.

Remark 2.9. $\mathrm{IC}^{\Sigma}_{+0}$ is an inferentially inconsistent (and so not subclassical) purely-inferential (and so both consistent and axiomatically-equivalent) extension of any purely-inferential Σ-logic C, in which case C is structurally complete iff it is inferentially inconsistent. In particular, any purely-inferential classical (and so inferentially consistent) Σ-logic is not structurally complete. □

3. Preliminary key advanced generic issues

3.1. **Equality determinants versus matrix hereditary simplicity.** Following the paradigm of the works [18] and [19], an *equality determinant for* a class of Σ-matrices M is any infinitary quantifier-free equality-free formula Φ of the first-order signature $L \triangleq (\Sigma \cup \{D\})$ (that is, any equality-free formula of the infinitary language $L_{\infty,0}$) with variables in Var_2 such that the infinitary universal sentence $\forall x_0 \forall x_1 (\Phi \leftrightarrow (x_0 \approx x_1))$ with equality is true in M, in which case Φ is an equality determinant for $\mathbf{I}(\mathbf{S}(\mathsf{M}))$ (cf. Lemma 3.3 of [23] for the "unitary" case discussed in Subsubsection 3.1.1). Then, a *canonical equality determinant for* M is any Σ-calculus ε of rank 2 such that $\bigwedge \varepsilon$ is an equality determinant for M. The main distinctive feature of Σ-matrices with equality determinant is as follows:

Lemma 3.1 (cf. Lemma 3.2 of [23] for the "unitary" case)**.** *Any Σ-matrix $\mathcal{A}$ with equality determinant Φ is simple, and so hereditarily so.*

Proof. Then, for any $\bar{a} \in \theta \in \mathrm{Con}(\mathcal{A})$, and all $\varphi \in \mathrm{Fm}^2_{\Sigma}$, we have $\varphi^{\mathfrak{A}}(a_0, a_0)\ \theta\ \varphi^{\mathfrak{A}}(a_0, a_1)$, in which case we get $(\varphi^{\mathfrak{A}}(a_0, a_0) \in D^{\mathcal{A}}) \Leftrightarrow (\varphi^{\mathfrak{A}}(a_0, a_1) \in D^{\mathcal{A}})$, and so $\mathcal{A} \models \Phi[x_i/a_i]_{i \in 2}$, for $\mathcal{A} \models \Phi[x_i/a_0]_{i \in 2}$, as $a_0 = a_0$ (in particular, $a_0 = a_1$, in which case $\theta = \Delta_A$, and so $\mathcal{A}$ is simple). □

Conversely, we have:

Theorem 3.2. *Every element of a class M of ⟨implicative⟩ Σ-matrices is [finitely] hereditarily simple iff M has a ({finitary/unary⟨/axiomatic⟩} canonical) equality determinant, in which case this is so for $\mathbf{IS}(\{\langle/\mathbf{PS}\rangle\})\mathsf{M}$, and so all elements of this class are hereditarily simple.*

Proof. The "if" part is by Lemma 3.1. Conversely, assume every element of M is finitely hereditarily simple. Consider any $\mathcal{A} \in \mathsf{M}$. Let $\varepsilon \triangleq \{\phi_i \vdash \phi_{1-i} \mid i \in 2, \bar{\phi} \in (\mathrm{Fm}^2_{\Sigma})^2, (\phi_0[x_1/x_0]) = (\phi_1[x_1/x_0])\}$. Clearly, $\mathcal{A} \models (\bigwedge \varepsilon)[x_i/a]_{i \in 2}$, for all $a \in A$, because every element of $\varepsilon[x_1/x_0]$ is a first-order tautology of the form $\zeta \vdash \zeta$, where $\zeta \in \mathrm{Fm}^2_{\Sigma}$. Conversely, consider any $\bar{a} \in (A^2 \setminus \Delta_A)$. Let $\mathcal{B}$ be the submatrix of

$\mathcal{A}$ generated by the finite set $\operatorname{img}\bar{a}$. Then, it, being finitely-generated is simple, in which case $\theta \triangleq \mathrm{Cg}^{\mathfrak{B}}(\bar{a}) \ni \bar{a} \notin \Delta_B$ is a non-diagonal congruence of $\mathfrak{B}$, and so $\theta \nsubseteq \theta^{\mathcal{B}}$. On the other hand, according to Mal'cev Principal Congruence Lemma [9] (cf. [4]), $\theta = \mathrm{Tr}(\nabla^{\mathfrak{A}}(\bar{a}) \cup \nabla^{\mathfrak{A}}(\bar{a})^{-1})$, where $\nabla^{\mathfrak{A}}(\bar{a}) \triangleq \{\langle \varphi^{\mathfrak{A}}[x_i/c_i; x_n/a_j]_{i\in n}\rangle_{j\in 2} \mid n \in \omega, \varphi \in \mathrm{Fm}_\Sigma^{n+1}, \bar{c} \in A^n\}$, in which case $\theta^{\mathcal{B}}$, being transitive and symmetric, does not include $\nabla^{\mathfrak{B}}(\bar{a})$, and so there are some $n \in \omega$, some $\varphi \in \mathrm{Fm}_\Sigma^{n+1}$ and some $\bar{c} \in B^n$ such that $\langle \varphi^{\mathfrak{B}}[x_n/a_j; x_i/c_i]_{i\in n}\rangle_{j\in 2} \notin \theta^{\mathcal{B}}$. Therefore, there is some $k \in 2$ such that $\varphi^{\mathfrak{B}}[x_n/a_k; x_i/c_i]_{i\in n} \in D^{\mathcal{B}} \not\ni \varphi^{\mathfrak{B}}[x_n/a_{1-k}; x_i/c_i]_{i\in n}$, while, as $\mathfrak{B}$ is generated by $\operatorname{img}\bar{a}$, for each $i \in n$, there is some $\psi_i \in \mathrm{Fm}_\Sigma^2$ such that $c_i = \psi_i^{\mathfrak{B}}[x_l/a_l]_{l\in 2}$. Then, $\phi_k^{\mathfrak{B}}[x_l/a_l]_{l\in 2} \in D^{\mathcal{B}} \not\ni \phi_{1-k}^{\mathfrak{B}}[x_l/a_l]_{l\in 2}$, where, for all $m \in 2$, $\phi_m \triangleq (\varphi[x_n/x_m; x_i/\psi_i]_{i\in n}) \in \mathrm{Fm}_\Sigma^2$. And what is more, $(\phi_0[x_1/x_0]) = (\varphi[x_i/(\psi_i[x_1/x_0])]_{i\in n}) = (\phi_1[x_1/x_0])$, in which case $(\phi_k \vdash \phi_{1-k}) \in \varepsilon$, and so $\mathcal{B} \not\models (\bigwedge\varepsilon)[x_l/a_l]_{l\in 2}$. Hence, $\mathcal{A} \not\models (\bigwedge\varepsilon)[x_l/a_l]_{l\in 2}$, for $\bigwedge\varepsilon$ is quantifier-free, and so ε is a unary (in particular, finitary) canonical equality determinant for M. ⟨Then, $\epsilon \triangleq \{\phi \sqsupset \psi \mid (\phi \vdash \psi) \in \varepsilon\}$ is an axiomatic canonical equality determinant for M.⟩ On the other hand, any $\Xi \subseteq \mathrm{Fm}_\Sigma^2$ is an axiomatic canonical equality determinant for a class of Σ-matrices K iff the universal infinitary strict Horn sentences with equality $\forall x_0 \forall x_1((\bigwedge\Xi) \to (x_0 \approx x_1))$ and $\forall x_0(\xi[x_1/x_0])$, where $\xi \in \Xi$, of the first-order signature $\Sigma \cup \{D\}$ are true in K. In this way, the well-known fact that model classes of universal infinitary ⟨strict Horn⟩ theories with equality are closed under **I** and **S** ⟨as well as **P**⟩ — cf., e.g., [10] — completes the argument. □

3.1.1. *Unitary equality determinants versus matrix non-diagonal partial automorphisms.* A *[partial] (strict) endomorphism of* a Σ-matrix $\mathcal{A}$ is any (strict) homomorphism from [a submatrix of] $\mathcal{A}$ to $\mathcal{A}$ ([injective ones being referred to as *partial automorphisms of* $\mathcal{A}$]).

A *unitary equality determinant for* a class M of Σ-matrices is any $\Upsilon \subseteq \mathrm{Fm}_\Sigma^1$ such that $\varepsilon_\Upsilon \triangleq \{(\upsilon[x_0/x_i]) \vdash (\upsilon[x_0/x_{1-i}]) \mid i \in 2, \upsilon \in \Upsilon\}$ is a (unary) canonical equality determinant for M. It is unitary equality determinants that are equality determinants in the sense of [18].

Theorem 3.3. *A Σ-matrix $\mathcal{A}$ has a unitary equality determinant iff it is (finitely) hereditarily simple and has no non-diagonal [injective] partial strict endomorphism.*

Proof. First, let Υ be a unitary equality determinant for $\mathcal{A}$, $\mathcal{B}$ a submatrix of $\mathcal{A}$ and $h \in \hom(\mathcal{B}, \mathcal{A})$ strict. Then, for every $b \in B$ and each $\upsilon \in \Upsilon$, we have $(\upsilon^{\mathfrak{A}}(b) = \upsilon^{\mathfrak{B}}(b) \in D^{\mathcal{A}}) \Leftrightarrow (\upsilon^{\mathfrak{B}}(b) \in D^{\mathcal{B}}) \Leftrightarrow (\upsilon^{\mathfrak{A}}(h(b)) = h(\upsilon^{\mathfrak{B}}(b)) \in D^{\mathcal{A}})$, in which case we get $h(b) = b$, and so h is diagonal. Thus, the "only if" part is by Lemma 3.1. Conversely, assume $\mathcal{A}$ has no non-diagonal partial automorphism and is finitely hereditarily simple, in which case, by Theorem 3.2, it has a unary canonical equality determinant ε. Consider any $\bar{a} \in A^2$ such that

$$(\varphi^{\mathfrak{A}}(a_0) \in D^{\mathcal{A}}) \Leftrightarrow (\varphi^{\mathfrak{A}}(a_1) \in D^{\mathcal{A}}), \tag{3.1}$$

for all $\varphi \in \mathrm{Fm}_\Sigma^1$. Let f be the carrier of the subalgebra of $\mathfrak{A}^2$ generated by $\{\bar{a}\}$, and, for each $i \in 2$, $\mathcal{B}_i$ the submatrix of $\mathcal{A}$ generated by $\{a_i\}$, in which case $B_i = \pi_i[f]$, for $\pi_i(\bar{a}) = a_i$, while $\pi_i \in \hom(\mathfrak{A}^2, \mathfrak{A})$. Consider any $i \in 2$ and any $\bar{b}, \bar{c} \in f$ such that $b_i \neq c_i$, in which case there are some $\phi, \psi \in \mathrm{Fm}_\Sigma^1$ such that $\bar{b} = \phi^{\mathfrak{A}^2}(\bar{a})$ and $\bar{c} = \psi^{\mathfrak{A}^2}(\bar{a})$ as well as some $(\xi \vdash \eta) \in \varepsilon$ such that $\xi^{\mathfrak{A}}(b_i, c_i) \in D^{\mathcal{A}} \not\ni \eta^{\mathfrak{A}}(b_i, c_i)$. Let $(\varpi|\zeta) \triangleq ((\xi|\eta)[x_0/\phi, x_1/\psi]) \in \mathrm{Fm}_\Sigma^1$, in which case $(\xi|\eta)^{\mathfrak{A}^2}(\bar{b}, \bar{c}) = (\varpi|\zeta)^{\mathfrak{A}^2}(\bar{a})$, and so $\varpi^{\mathfrak{A}}(a_i) \in D^{\mathcal{A}} \not\ni \zeta^{\mathfrak{A}}(a_i)$. Hence, by (3.1), $\xi^{\mathfrak{A}}(b_{1-i}, c_{1-i}) = \varpi^{\mathfrak{A}}(a_{1-i}) \in D^{\mathcal{A}} \not\ni \zeta^{\mathfrak{A}}(a_{1-i}) = \eta^{\mathfrak{A}}(b_{1-i}, c_{1-i})$, in which case $b_{1-i} \neq c_{1-i}$, and so $f : B_0 \to B_1$ is injective. Therefore, f, being a subalgebra of $\mathfrak{A}^2$, is an embedding of $\mathfrak{B}_0$ into $\mathfrak{A}$, in which case, by (3.1), f is an embedding of $\mathcal{B}_0$ into $\mathcal{A}$, and so a partial automorphism of $\mathcal{A}$. Thus, f is diagonal, in which case $a_1 = f(a_0) = a_0$, so Fm^1_Σ is a unitary equality determinant for $\mathcal{A}$. □

Clearly, any consistent truth-non-empty two-valued (in particular, classical) Σ-matrix $\mathcal{A}$ is both false- and truth-singular, in which case its characteristic relation is diagonal, and so $\{x_0\}$ is an equality determinant for $\mathcal{A}$.

3.2. **Disjunctivity.**

3.2.1. *Disjunctivity versus multiplicativity.* A Σ-logic C is said to be $\veebar$-*(singularly-)multiplicative*, provided, for all $X \subseteq \mathrm{Fm}^\omega_\Sigma$ and all $\phi, \psi \in \mathrm{Fm}^\omega_\Sigma$, it holds that $(\veebar[C(X \cup \{\phi\}) \times \{\psi\}]) \subseteq C(X \cup \{\phi \veebar \psi\})$.

Lemma 3.4. *Any Σ-logic C is $\veebar$-disjunctive iff it is both weakly $\veebar$-disjunctive and $\veebar$-multiplicative as well as satisfies both* (2.3) *and* (2.4)*.*

Proof. The "only if" part is immediate. Conversely, assume C is both weakly $\veebar$-disjunctive and $\veebar$-multiplicative as well as satisfies both (2.3) and (2.4). Consider any $X \subseteq \mathrm{Fm}^\omega_\Sigma$, any $\phi, \psi \in \mathrm{Fm}^\omega_\Sigma$ and any $\varphi \in (C(X \cup \{\phi\}) \cap C(X \cup \{\psi\}))$. Then, by the $\veebar$-multiplicativity of C and (2.3), we have $(\psi \veebar \varphi) \in C(\varphi \veebar \psi) \subseteq C(X \cup \{\phi \veebar \psi\})$. Likewise, by the $\veebar$-multiplicativity of C and (2.4), we have $\varphi \in C(\varphi \veebar \varphi) \subseteq C(X \cup \{\psi \veebar \varphi\})$. In this way, we eventually get $\varphi \in C(X \cup \{\phi \veebar \psi\})$. □

3.2.1.1. Implicativity versus intrinsic disjunctivity.

Theorem 3.5. *Let C be a weakly $\sqsupset$-implicative Σ-logic and $\veebar \triangleq \uplus_\sqsupset$. Then, the following hold:*

(i) *C is both weakly $\veebar$-disjunctive and $\veebar$-multiplicative;*
(ii) *C is $\sqsupset$-implicative iff it is $\veebar$-disjunctive iff it satisfies* (2.3)*.*

Proof. (i) First, (2.2) with $i = 0$ is by DT and (2.7). Likewise, (2.2) with $i = 1$ is by (2.6) and (2.7). Now, consider any $X \subseteq \mathrm{Fm}^\omega_\Sigma$ and any $\phi, \psi, \varphi \in \mathrm{Fm}^\omega_\Sigma$. Then, by DT and (2.7), we have $((\psi \in C(X \cup \{\phi\}) \Rightarrow ((\phi \sqsupset \varphi) \in C(X \cup \{\psi \sqsupset \varphi\})$, applying which twice, the second time being with $(\psi \sqsupset \varphi)|(\phi \sqsupset \varphi)$ instead of $\phi|\psi$, respectively, we conclude that C is $\veebar$-multiplicative.

(ii) Assume C is $\sqsupset$-implicative. Then, $((x_0 \veebar x_0) \sqsupset x_0) = ((2.9)[x_1/x_0])$ is satisfied in C, for this is structural, and so is (2.4), in view of (2.7). Furthermore, by (2.7), we have $x_0 \in C(\{x_0 \veebar x_1, x_0 \sqsupset x_1, x_1 \sqsupset x_0\})$, in which case, by DT, we get $((x_0 \sqsupset x_1) \sqsupset x_0) \in C(\{x_0 \veebar x_1, x_1 \sqsupset x_0\})$, and so, by (2.7) and (2.9), we eventually get $x_0 \in C(\{x_0 \veebar x_1, x_1 \sqsupset x_0\})$ (in particular, by DT, (2.3) is satisfied in C). Then, Lemma 3.4, (i) and (2.8) complete the argument. □

3.2.2. *Disjunctive consistent finitely-generated models of finitely-valued weakly disjunctive logics.*

Lemma 3.6. $\mathbf{H}(\mathbf{H}^{-1}(\mathsf{M})) \subseteq \mathbf{H}^{-1}(\mathbf{H}(\mathsf{M}))$*, for any class of Σ-matrices* M*.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be Σ-matrices, $\mathcal{C} \in \mathsf{M}$ and $(h|g) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{B}, \mathcal{C}|\mathcal{A})$. Then, by Remark 2.6(i), $(\ker(h|g)) \in \mathrm{Con}(\mathcal{B})$, in which case $(\ker(h|g)) \subseteq \theta \triangleq \eth(\mathcal{B}) \in \mathrm{Con}(\mathcal{B})$, and so, by the Homomorphism Theorem, $(\nu_\theta \circ (h|g)^{-1}) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{C}|\mathcal{A}, \mathcal{B}/\theta)$. □

Lemma 3.7 (cf. the proof of Lemma 2.7 of [23])**.** *Let M be a (finite) class of (finite) Σ-matrices and $\mathcal{A}$ a [non-]simple denumerably-generated (more specifically, finite{ly-generated}) model of the logic of M. (Suppose $\mathcal{A}$ is {generated by a set} of cardinality $n \in \omega$.) Then, there are some (finite) set I (of cardinality $\leqslant \sum_{\mathcal{B} \in \mathsf{M}} n^{|B|}$), some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ and some its subdirect product in $\mathbf{H}^{-1}(\mathcal{A}[/\eth(\mathcal{A})])$.*

Lemma 3.8. *Let* M *be a class of weakly* $\veebar$*-disjunctive* Σ*-matrices,* I *a finite set,* $\overline{\mathcal{C}} \in \mathsf{M}^I$*, and* $\mathcal{D}$ *a consistent* $\veebar$*-disjunctive submatrix of* $\prod \overline{\mathcal{C}}$*. Then, there is some* $i \in I$ *such that* $(\pi_i \upharpoonright D) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}, \mathcal{C}_i)$*.*

Proof. By contradiction. For suppose that, for every $i \in I$, $(\pi_i \upharpoonright D) \notin \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}, \mathcal{C}_i)$, in which case $D^{\mathcal{D}} \subsetneq (\pi_i \upharpoonright D)^{-1}[D^{\mathcal{C}_i}] = (D \cap \pi_i^{-1}[D^{\mathcal{C}_i}])$, for $(\pi_i \upharpoonright D) \in \hom(\mathcal{D}, \mathcal{C}_i)$ is surjective, and so there is some $a_i \in (D \setminus D^{\mathcal{D}})$ such that $\pi_i(a_i) \in D^{\mathcal{C}_i}$. By induction on the cardinality of any $J \subseteq I$, let us prove that there is some $b \in (D \setminus D^{\mathcal{D}})$ such that $\pi_j(b) \in D^{\mathcal{C}_j}$, for all $j \in J$, as follows. In case $J = \varnothing$, take any $b \in (D \setminus D^{\mathcal{D}}) \neq \varnothing$, for $\mathcal{D}$ is consistent. Otherwise, take any $j \in J$, in which case $K \triangleq (J \setminus \{j\}) \subseteq I$, while $|K| < |J|$, so, by the induction hypothesis, there is some $c \in (D \setminus D^{\mathcal{D}})$ such that $\pi_k(c) \in D^{\mathcal{C}_k}$, for all $k \in K$. Then, by the $\veebar$-disjunctivity of $\mathcal{D}$, $b \triangleq (c \veebar^{\mathfrak{D}} a_j) \in (D \setminus D^{\mathcal{D}})$, while $\pi_i(b) \in D^{\mathcal{C}_i}$, for all $i \in J = (K \cup \{j\})$, because $(\pi_i \upharpoonright D) \in \hom(\mathfrak{D}, \mathfrak{C}_i)$, while $\mathcal{C}_i$ is weakly $\veebar$-disjunctive. In particular, when $J = I$, there is some $b \in (D \setminus D^{\mathcal{D}})$ such that $\pi_i(b) \in D^{\mathcal{C}_i}$, for all $i \in I$. This contradicts to the fact that $D^{\mathcal{D}} = (D \cap \bigcap_{i \in I} \pi_i^{-1}[D^{\mathcal{C}_i}])$, as required. □

By Lemmas 3.6, 3.7, 3.8 and Remark 2.8(ii), we immediately have:

Theorem 3.9. *Let* M *be a finite class of finite weakly* $\veebar$*-disjunctive* Σ*-matrices,* C *the logic of* M *and* $\mathcal{A}$ *a finite[ly-generated] consistent* $\veebar$*-disjunctive model of* C*. Then,* $\mathcal{A} \in \mathbf{H}^{-1}(\mathbf{H}(\mathbf{S}_*(\mathsf{M})))$*.*

3.2.2.1. Theorems of weakly disjunctive finitely-valued logics versus truth-empty submatrices of defining matrices.

Corollary 3.10. *Let* C *be a* Σ*-logic. (Suppose it is defined by a finite class* M *of finite [weakly* $\veebar$*-disjunctive]* Σ*-matrices.) Then, (i)⇔(ii)⇔(iii)(⇔(iv)), where:*

(i) *C is purely-inferential;*
(ii) *C has a truth-empty model;*
(iii) *C has a one-valued truth-empty model;*
(iv) $\mathbf{P}^{\mathrm{SD}}_{\omega[\cap 0]}(\mathbf{S}_*(\mathsf{M}))[\cup \mathbf{S}_*(\mathsf{M})]$ *has a truth-empty element.*

Proof. First, (ii)⇒(i) is immediate. The converse is by the fact that, by the structurality of C, $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, C(\varnothing) \rangle$ is a model of C.

Next, (ii) is a particular case of (iii). Conversely, let $\mathcal{A} \in \mathrm{Mod}(C)$ be truth-empty. Then, $\chi^{\mathcal{A}}$ is singular, in which case $\theta^{\mathcal{A}} = A^2 \in \mathrm{Con}(\mathfrak{A})$, and so, by (2.14) and Remark 2.8(ii)(b), $(\mathcal{A}/\theta^{\mathcal{A}}) \in \mathrm{Mod}(C)$ is both one-valued and truth-empty.

(Finally, (iv)⇒(ii) is by (2.14). Conversely, (iii)⇒(iv) is by Remark 2.8(ii)(b) and Lemma 3.7 [resp., Theorem 3.9 as well as the consistency and $\veebar$-disjunctivity of truth-empty Σ-matrices].) □

3.2.3. *Non-paraconsistency versus Resolution.* Given any Σ-logic C, by C^{R} we denote the extension of C relatively axiomatized by the *Resolution* rule (cf. [27]):

$$\{x_0 \veebar x_1, \wr x_0 \veebar x_1\} \vdash x_1. \tag{3.2}$$

Applying Lemma 3.4 and (2.4) to (2.10) twice, we have:

Lemma 3.11. (3.2) *is satisfied in any* $\veebar$*-disjunctive non-*$\wr$*-paraconsistent* Σ*-logic.*

Theorem 3.12. *Let* M *be a finite class of finite* $\veebar$*-disjunctive* Σ*-matrices and* C *the logic of* M*. Then,* C^{R} *is defined by the class* S *of all non-*$\wr$*-paraconsistent elements of* $\mathbf{S}_*(\mathsf{M})$*, and so is* $\veebar$*-disjunctive but is not* $\wr$*-paraconsistent.*

Proof. Then, C is $\veebar$-disjunctive, while the logic of S is a both finitary, $\veebar$-disjunctive (in view of Remark 2.8(ii)(a)) and non-$\wr$-paraconsistent extension of C, and so an extension of C^{R}, in view of Lemma 3.11. Conversely, consider any $n \in (\omega \setminus 1)$, any $\Gamma \subseteq \mathrm{Fm}^n_\Sigma$ and any $\varphi \in (\mathrm{Fm}^n_\Sigma \setminus C^{\mathrm{R}}(\Gamma))$, in which case, by (2.13) with $\alpha = n$, $\varphi \notin C(\Gamma) = \mathrm{Cn}^\omega_{\mathsf{M}}(\Gamma) \supseteq \mathrm{Cn}^n_{\mathsf{M}}(\Gamma)$, and so $\mathcal{T} \triangleq \{T \in \mathcal{B}^n_{\mathsf{M}} \mid \Gamma \subseteq T \not\ni \varphi\} \neq \varnothing$. Then, since n as well as both M and all elements of it are finite, the class $\{\langle \mathcal{A}, h\rangle \mid \mathcal{A} \in \mathsf{M}, h \in \hom(\mathfrak{Fm}^n_\Sigma, \mathfrak{A})\}$ is finite, in which case the set $\mathcal{B}^n_{\mathsf{M}}$ is finite, and so is $\mathcal{T} \subseteq \mathcal{B}^n_{\mathsf{M}}$. Let $m \triangleq |\mathcal{T}| \in (\omega \setminus 1)$ and $\overline{T} : m \to \mathcal{T}$ bijective, in which case, for each $i \in m$, there is some $\mathcal{A}_i \in \mathsf{M}$ and some $h_i \in \hom(\mathfrak{Fm}^n_\Sigma, \mathfrak{A}_i)$ such that $\Gamma \subseteq T_i = h_i^{-1}[D^{\mathcal{A}_i}] \not\ni \varphi$, and so $B_i \triangleq (\mathrm{img}\, h_i)$ forms a subalgebra of $\mathfrak{A}_i$, while $\mathcal{B}_i \triangleq (\mathcal{A}_i \upharpoonright B_i) \in \mathbf{S}(\mathsf{M})$, whereas $h_i^{-1}[D^{\mathcal{B}_i}] = T_i$ (in particular, $\mathcal{B}_i$ is consistent, for $h_i(\varphi) \in (B_i \setminus D^{\mathcal{B}_i})$), as well as $h_i \in \hom(\mathfrak{Fm}^n_\Sigma, \mathfrak{B}_i)$ (In particular, $T_i \in \mathcal{B}^n_{\mathbf{S}_*(\mathsf{M})}$). We prove, by contradiction, that, for some $i \in m$, $\mathcal{B}_i$ is not $\wr$-paraconsistent. For suppose each $\mathcal{B}_i$, where $i \in m$, is $\wr$-paraconsistent. By induction on any $j \in (m+1)$, we set $\Xi_j \triangleq (\{\varphi\} | \{\wr^k \psi \veebar \phi \mid k \in 2, \psi \in T_{j-1} \ni \wr\psi, \phi \in \Xi_{j-1}\}) \subseteq \mathrm{Fm}^n_\Sigma$, whenever $j = | \neq 0$, respectively, and prove that

$$\varphi \in C^{\mathrm{R}}(\Xi_j), \tag{3.3}$$

$$\Xi_j \subseteq (C(T_i) \cap C(\Xi_i)), \tag{3.4}$$

for all $i \in j$. The case, when $j = 0 = \varnothing$, is evident. Otherwise, $(j-1) \in (m \cap j)$, in which case $\mathcal{B}_{j-1}$ is $\sim$-paraconsistent, and so there is some $\psi \in T_{j-1}$ such that $\wr\psi \in T_{j-1}$. In particular, for each $\phi \in \Xi_{j-1}$ and every $k \in 2$, $(\wr^k\psi \veebar \phi) \in \Xi_j$, in which case, by (3.2)$[x_0/\psi, x_1/\phi]$ and the structurality of C^{R}, $\phi \in C^{\mathrm{R}}(\Xi_j)$, and so, by the induction hypothesis, $\varphi \in C^{\mathrm{R}}(\Xi_{j-1}) \subseteq C^{\mathrm{R}}(\Xi_j)$. Thus, (3.3) holds. Likewise, by the $\veebar$-disjunctivity of C, for each $\phi \in \Xi_{j-1}$, every $k \in 2$ and all $\psi \in T_{j-1}$ such that $\wr\psi \in T_{j-1}$, we have $(\wr^k\psi \veebar \phi) \in (C(\Xi_{j-1}) \cap C(T_{j-1}))$ (in particular, (3.4) with $i = (j-1)$ holds), and so, by the induction hypothesis as well as (3.4) with $i = (j-1)$, we get (3.4), for all $i \in (j-1)$. Thus, (3.4) holds, for all $i \in (\{j-1\} \cup (j-1)) = j$. In this way, by (3.3) with $j = m$, we have $\Xi_m \nsubseteq C^{\mathrm{R}}(\Gamma) \supseteq C(\Gamma) = \mathrm{Cn}^\omega_{\mathsf{M}}(\Gamma)$, in which case, by (2.13) with $\alpha = n$, we get $\Xi_m \nsubseteq \mathrm{Cn}^n_{\mathsf{M}}(\Gamma)$, and so there is some $T \in \mathcal{B}^n_{\mathsf{M}}$ such that $\Gamma \subseteq T \not\supseteq \Xi_m$. In that case, if T contained φ, that is, included Ξ_0, then, by (3.4) with $j = m$ and $i = 0 \in m$, for $m \neq 0$, we would have $\Xi_m \subseteq C(T)$, and so, by (2.13) with $\alpha = n$, would get $\Xi_m \subseteq \mathrm{Cn}^n_{\mathsf{M}}(T) = T$. Therefore, $\varphi \notin T$, in which case $T \in \mathcal{T}$, and so $T = T_l$, for some $l \in m$. Hence, by (3.4) with $j = m$ and $i = l$, we have $\Xi_m \subseteq C(T)$, in which case, by (2.13) with $\alpha = n$, we get $\Xi_m \subseteq \mathrm{Cn}^n_{\mathsf{M}}(T) = T$, and so this contradiction shows that there is some $i \in m$, such that $\mathcal{B}_i$ is not $\wr$-paraconsistent. In this way, $\mathcal{B}_i \in \mathsf{S}$, in which case $\varphi \notin \mathrm{Cn}^n_{\mathcal{B}_i}(\Gamma) \supseteq \mathrm{Cn}^n_{\mathsf{S}}(\Gamma)$, and so, by (2.13) with $\alpha = n$, $\varphi \notin \mathrm{Cn}^\omega_{\mathsf{S}}(\Gamma)$, as required, for $\wp_\omega(\mathrm{Fm}^\omega_\Sigma) \subseteq \bigcup_{n \in (\omega \setminus 1)} \wp(\mathrm{Fm}^n_\Sigma)$. ☐

3.3. Some peculiarities of false-singular matrices.

3.3.1. *Subdirect products of consistent submatrices of weakly conjunctive matrices.*

Lemma 3.13. *Let $\mathcal{A}$ be a false-singular weakly $\diamond$-conjunctive Σ-matrix, $f \in (A \setminus D^{\mathcal{A}})$, I a finite set, $\overline{\mathcal{B}} \in \mathbf{S}_*(\mathcal{A})^I$ and $\mathcal{D}$ a subdirect product of it. Then, $(I \times \{f\}) \in D$.*

Proof. By induction on the cardinality of any $J \subseteq I$, let us prove that there is some $a \in D$ including $(J \times \{f\})$. First, when $J = \varnothing$, take any $a \in D \neq \varnothing$, in which case $(J \times \{f\}) = \varnothing \subseteq a$. Now, assume $J \neq \varnothing$. Take any $j \in J \subseteq I$, in which case $K \triangleq (J \setminus \{j\}) \subseteq I$, while $|K| < |J|$, and so, as $\mathcal{B}_j$ is a consistent submatrix of the false-singular Σ-matrix $\mathcal{A}$, we have $f \in B_j = \pi_j[D]$. Hence, there is some $b \in D$ such that $\pi_j(b) = f$, while, by induction hypothesis, there is some $c \in D$ including $(K \times \{f\})$. Therefore, since $J = (K \cup \{j\})$, while $\mathcal{A}$ is both weakly $\diamond$-conjunctive and false-singular, we have $D \ni a \triangleq (c \diamond^{\mathfrak{D}} b) \supseteq (J \times \{f\})$. Thus, when $J = I$, we eventually get $D \ni (I \times \{f\})$, as required. ☐

3.3.2. *Models of weakly implicative logics.*

Lemma 3.14. *Let $\mathcal{A}$ be a false-singular Σ-matrix. Suppose* (2.5)*,* (2.6) *and* (2.7) *are true in $\mathcal{A}$. Then, $\mathcal{A}$ is $\sqsupset$-implicative. In particular, any false-singular Σ-matrix is $\sqsupset$-implicative iff its logic is [weakly] so.*

Proof. Then, for all $a, b \in (A \setminus D^{\mathcal{A}})$, we have $a = b$, in which case, by (2.5), we get $(a \sqsupset^{\mathfrak{A}} b) = (a \sqsupset^{\mathfrak{A}} a) \in D^{\mathcal{A}}$, and so (2.6) and (2.7) complete the argument. □

3.4. **Logic versus model congruences.**

Lemma 3.15. *Let C be a Σ-logic, $\theta \in \mathrm{Con}(C)$, $\mathcal{A} \in \mathrm{Mod}(C)$ and $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$. Then, $h[\theta] \subseteq \partial(\mathcal{A})$.*

Proof. Then, $\vartheta \triangleq (\bigcup\{g[\theta] \mid g \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})\})$ is symmetric, for θ is so. And what is more, since $\theta \subseteq \equiv^{\omega}_{C}$, while $\mathcal{A} \in \mathrm{Mod}(C)$, $\vartheta \subseteq \theta^{\mathcal{A}}$. Next, consider any $a \in A$. Let $g \triangleq [x_k/a]_{k\in\omega} \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$. Then, since $\langle x_0, x_0\rangle \in \theta$, $\langle a, a\rangle = g(\langle x_0, x_0\rangle) \in g[\theta] \subseteq \vartheta$, and so $\Delta_A \subseteq \vartheta$. Now, consider any $\varsigma \in \Sigma$ of arity $n \in \omega$, any $i \in n$, any $\langle a, b\rangle \in \vartheta$ and any $\bar{c} \in A^{n-1}$. Then, there are some $\langle \phi, \psi\rangle \in \theta$ and some $f \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$ such that $a = f(\phi)$ and $b = f(\psi)$. Let $V \triangleq (\mathrm{Var}(\phi) \cup \mathrm{Var}(\psi) \cup \{x_i\}) \in \wp_{\omega}(\mathrm{Var}_{\omega})$, in which case $|\mathrm{Var}_{\omega} \setminus V| = \omega \geqslant (n-1)$, for $|\mathrm{Var}_{\omega}| = \omega$ is infinite, and so there is some injective $\bar{v} \in (\mathrm{Var}_{\omega} \setminus V)^{n-1}$. Let $\varphi \triangleq (\varsigma(\bar{x}_n)[x_j/v_j; x_k/v_{k-1}]_{j\in i; k\in(n\setminus(i+1))}) \in \mathrm{Fm}^{\omega}_{\Sigma}$ and $g \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$ extend $(f \upharpoonright (\mathrm{Var}_{\omega} \setminus (\mathrm{img}\,\bar{v}))) \cup (\bar{c} \circ \bar{v}^{-1})$, in which case $\langle \varphi[x_i/\phi], \varphi[x_i/\psi]\rangle \in \theta$, so $\langle \varphi^{\mathfrak{A}}[x_i/a; v_l/c_l]_{l\in(n-1)}, \varphi^{\mathfrak{A}}[x_i/b; v_l/c_l]_{l\in(n-1)}\rangle = g(\langle \varphi[x_i/\phi], \varphi[x_i/\psi]\rangle) \in g[\theta] \subseteq \vartheta$. Thus, unary algebraic operations of $\mathfrak{A}$ are ϑ-monotonic. Therefore, $\eta \triangleq \mathrm{Tr}(\vartheta)$ is a congruence of $\mathfrak{A}$. And what is more, $\theta^{\mathcal{A}} \supseteq \vartheta$, being transitive, includes η, in which case $\eta \in \mathrm{Con}(\mathcal{A})$, and so $h[\theta] \subseteq \vartheta \subseteq \eta \subseteq \partial(\mathcal{A})$. □

3.4.1. *Simple models versus intrinsic varieties.* As a particular case of Lemma 3.15, we first have (from now on, we follow Definition 2.2 tacitly):

Corollary 3.16. *Let C be a Σ-logic. Then, $\pi_0[\mathrm{Mod}^*(C)] \subseteq \mathrm{IV}(C)$.*

Corollary 3.17. *Let C be a Σ-logic. Then, $\partial(C)$ is fully-invariant. In particular, $\partial(C) = \theta^{\omega}_{\mathrm{IV}(C)}$.*

Proof. Consider any $\sigma \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{Fm}^{\omega}_{\Sigma})$ and any $T \in (\mathrm{img}\, C)$, in which case, by the structurality of C, $\mathcal{A}_T \triangleq \langle \mathfrak{Fm}^{\omega}_{\Sigma}, T\rangle \in \mathrm{Mod}(C)$, so, by Lemma 3.15, $\sigma[\partial(C)] \subseteq \partial(\mathcal{A}_T)$. Then, $\sigma[\partial(C)] \subseteq \theta \triangleq (\mathrm{Eq}^{\omega}_{\Sigma} \cap \bigcap\{\partial(\mathcal{A}_T) \mid T \in (\mathrm{img}\, C)\}) \subseteq (\mathrm{Eq}^{\omega}_{\Sigma} \cap \bigcap\{\theta^{\mathcal{A}_T} \mid T \in (\mathrm{img}\, C)\} = \equiv^{\omega}_{C}$. Moreover, for each $T \in (\mathrm{img}\, C)$, $\partial(\mathcal{A}_T) \in \mathrm{Con}(\mathfrak{Fm}^{\omega}_{\Sigma})$, in which case $\theta \in \mathrm{Con}(\mathfrak{Fm}^{\omega}_{\Sigma})$, and so $\sigma[\partial(C)] \subseteq \theta \subseteq \partial(C)$. □

Lemma 3.18. *Let M be a class of Σ-matrices, $\mathsf{K} \triangleq \pi_0[\mathsf{M}]$ and C the logic of M. Then, $\theta^{\omega}_{\mathsf{K}} \subseteq \equiv^{\omega}_{C}$, in which case $\theta^{\omega}_{\mathsf{K}} \subseteq \partial(C)$, and so $\mathrm{IV}(C) \subseteq \mathbf{V}(\mathsf{K})$.*

Proof. Then, for any $\langle \phi, \psi\rangle \in \theta^{\omega}_{\mathsf{K}}$, $\mathcal{A} \in \mathsf{M}$ and $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$, $\mathfrak{A} \in \mathsf{K}$, in which case $\langle h(\phi), h(\psi)\rangle \in \Delta_A \subseteq \theta^{\mathcal{A}}$, and so $\phi \equiv^{\omega}_{C} \psi$. □

By Corollary 3.16 and Lemma 3.18, we then have:

Corollary 3.19. *Let M be a class of Σ-matrices, $\mathsf{K} \triangleq \pi_0[\mathsf{M}]$ and C the logic of M. Then, $\pi_0[\mathrm{Mod}^*(C)] \subseteq \mathbf{V}(\mathsf{K})$.*

Theorem 3.20. *Let M be a class of simple Σ-matrices, $\mathsf{K} \triangleq \pi_0[\mathsf{M}]$ and C the logic of M. Then, $\mathrm{IV}(C) = \mathbf{V}(\mathsf{K})$.*

4. Self-extensional logics versus simple matrices

Theorem 4.1. *Let C be a Σ-logic and $\mathsf{V} \triangleq \mathrm{IV}(C)$ (as well as M a class of simple Σ-matrices, $\mathsf{K} \triangleq \pi_0[\mathsf{M}]$ and $\alpha \triangleq ([1\cup](\omega \cap \bigcup\{|A| \mid \mathcal{A} \in \mathsf{M}\}))$). (Suppose C is defined by M.) Then, (i)⇔(ii)⇔(iii)(⇒(iv)⇒(v)⇒)(vi)⇒(i), where:*

(i) *C is self-extensional;*
(ii) *$\equiv^\omega_C \subseteq \theta^\omega_\mathsf{V}$;*
(iii) *$\equiv^\omega_C = \theta^\omega_\mathsf{V}$;*
(iv) *for all distinct $a, b \in F^\alpha_\mathsf{V}$, there are some $\mathcal{A} \in \mathsf{M}$ and some $h \in \hom(\mathfrak{F}^\alpha_\mathsf{V}, \mathfrak{A})$ such that $\chi^\mathcal{A}(h(a)) \neq \chi^\mathcal{A}(h(b))$;*
(v) *there is some class C of Σ-algebras such that $\mathsf{K} \subseteq \mathbf{V}(\mathsf{C})$ and, for each $\mathfrak{A} \in \mathsf{C}$ and all distinct $a, b \in A$, there are some $\mathcal{B} \in \mathsf{M}$ and some $h \in \hom(\mathfrak{A}, \mathfrak{B})$ such that $\chi^\mathcal{B}(h(a)) \neq \chi^\mathcal{B}(h(b))$;*
(vi) *there is some $\mathsf{S} \subseteq \mathrm{Mod}(C)$ such that $\mathsf{V} \subseteq \mathbf{V}(\pi_0[\mathsf{S}])$ and, for each $\mathcal{A} \in \mathsf{S}$, it holds that $(A^2 \cap \bigcap\{\theta^\mathcal{B} \mid \mathcal{B} \in \mathsf{S}, \mathfrak{B} = \mathfrak{A}\}) \subseteq \Delta_A$.*

Proof. In that case, by Corollary 3.17 (and Theorem 3.20), $\supsetneq(C) = \theta^\omega_\mathsf{V}$ (as well as $\mathsf{V} = \mathbf{V}(\mathsf{K})$, and so $\theta^\omega_\mathsf{V} = \theta^\omega_\mathsf{K}$). Then, (i)⇔(iii) is immediate, while (ii) is a particular case of (iii), whereas the converse is by the inclusion $\supsetneq(C) \subseteq \equiv^\omega_C$.

(Next, assume (iii) holds. Then, $\theta^{\alpha'} \triangleq \equiv^{\alpha'}_C = \theta^{\alpha'}_\mathsf{K} = \theta^{\alpha'}_\mathsf{V} \in \mathrm{Con}(\mathfrak{Fm}^{\alpha'}_\Sigma)$, for all Σ-ranks α'. Furthermore, consider any distinct $a, b \in F^\alpha_\mathsf{V}$. Then, there are some $\phi, \psi \in \mathrm{Fm}^\alpha_\Sigma$ such that $\nu_{\theta^\alpha}(\phi) = a \neq b = \nu_{\theta^\alpha}(\phi)$, in which case, by (2.13), $\mathrm{Cn}^\alpha_\mathsf{M}(\phi) \neq \mathrm{Cn}^\alpha_\mathsf{M}(\psi)$, and so there are some $\mathcal{A} \in \mathsf{M}$ and some $g \in \hom(\mathfrak{Fm}^\alpha_\Sigma, \mathfrak{A})$ such that $\chi^\mathcal{A}(g(\phi)) \neq \chi^\mathcal{A}(g(\phi))$. In that case, $\theta^\alpha \subseteq (\ker g)$, and so, by the Homomorphism Theorem, $h \triangleq (g \circ \nu^{-1}_{\theta^\alpha}) \in \hom(\mathfrak{F}^\alpha_\mathsf{V}, \mathfrak{A})$. Then, $h(a/b) = g(\phi/\psi)$, in which case $\chi^\mathcal{A}(h(a)) \neq \chi^\mathcal{A}(h(b))$, and so (iv) holds.

Now, assume (iv) holds. Consider any $\mathfrak{A} \in \mathsf{K}$ and the following cases:

- $|A| \leqslant \alpha$. Let $h \in \hom(\mathfrak{Fm}^\alpha_\Sigma, \mathfrak{A})$ extend any surjection from Var_α onto A, in which case it is surjective, while $\theta \triangleq \theta^\alpha_\mathsf{V} = \theta^\alpha_\mathsf{K} \subseteq (\ker h)$, and so, by the Homomorphism Theorem, $g \triangleq (h \circ \nu^{-1}_\theta) \in \hom(\mathfrak{F}^\alpha_\mathsf{V}, \mathfrak{A})$ is surjective. Thus, $\mathfrak{A} \in \mathbf{V}(\mathfrak{F}^\alpha_\mathsf{V})$.
- $|A| \nleqslant \alpha$. Then, $\alpha = \omega$. Consider any Σ-identity $\phi \approx \psi$ true in $\mathfrak{F}^\omega_\mathsf{V}$ and any $h \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{A})$, in which case, we have $\theta \triangleq \theta^\omega_\mathsf{V} = \theta^\omega_\mathsf{K} \subseteq (\ker h)$, and so, since $\nu_\theta \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{F}^\omega_\mathsf{V})$, we get $\langle\phi, \psi\rangle \in (\ker \nu_\theta) \subseteq (\ker h)$. Thus, $\mathfrak{A} \in \mathbf{V}(\mathfrak{F}^\alpha_\mathsf{V})$.

In this way, (v) with $\mathsf{C} \triangleq \{\mathfrak{F}^\alpha_\mathsf{V}\}$ holds.

Further, assume (v) holds. Let $\mathsf{C}' \triangleq \{\mathfrak{A} \in \mathsf{C} \mid |A| > 1\}$ and $\mathsf{S} \triangleq \{\langle\mathfrak{A}, h^{-1}[D^\mathcal{B}]\rangle \mid \mathfrak{A} \in \mathsf{C}', \mathcal{B} \in \mathsf{M}, h \in \hom(\mathfrak{A}, \mathfrak{B})\}$. Then, for all $\mathfrak{A} \in \mathsf{C}'$, each $\mathcal{B} \in \mathsf{M}$ and every $h \in \hom(\mathfrak{A}, \mathfrak{B})$, h is a strict homomorphism from $\mathcal{C} \triangleq \langle\mathfrak{A}, h^{-1}[D^\mathcal{B}]\rangle$ to $\mathcal{B}$, in which case, by (2.14), $\mathcal{C} \in \mathrm{Mod}(C)$, and so $\mathsf{S} \subseteq \mathrm{Mod}(C)$, while $\chi^\mathcal{C} = (h \circ \chi^\mathcal{B})$, whereas $\pi_0[\mathsf{S}] = \mathsf{C}'$ generates the variety $\mathbf{V}(\mathsf{C})$. In this way, (vi) holds.)

Finally, assume (vi) holds. Consider any $\phi, \psi \in \mathrm{Fm}^\omega_\Sigma$ such that $\phi \equiv^\omega_C \psi$, any $\mathcal{A} \in \mathsf{S}$ and any $h \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{A})$. Then, for each $\mathcal{B} \in \mathsf{S}$ with $\mathfrak{B} = \mathfrak{A}$, $h(\phi)\ \theta^\mathcal{B}\ h(\psi)$, in which case $h(\phi) = h(\psi)$, so $\mathfrak{A} \models (\phi \approx \psi)$. Thus, $\mathsf{V} \subseteq \mathbf{V}(\pi_0[\mathsf{S}]) \models (\phi \approx \psi)$, so (ii) holds. □

When both M and all elements of it are finite, α is finite, in which case $\mathfrak{F}^\alpha_\mathsf{V}$ is finite and can be found effectively, and so, taking (2.14) and Remark 2.6(iv) into account, the item (iv) of Theorem 4.1 yields an effective procedure of checking the self-extensionality of any logic defined by a finite class of finite matrices. However, its computational complexity may be too large to count it *practically* applicable. For instance, in the unitary n-valued case, where $n \in (\omega \setminus 1)$, the upper limit n^{n^n} of $|F^\alpha_\mathsf{V}|$ as well as the predetermined computational complexity $n^{n^{n^n}}$ of the procedure involved become too large even in the three-/four-valued case. And, though, in the two-valued case, this limit — 16 — as well as the respective complexity — $2^{16} = 65536$ — are reasonably acceptable, this is no longer matter in view of:

Example 4.2. Let $\mathcal{A}$ be a Σ-matrix. Suppose it is both false- and truth-singular (in particular, two-valued as well as both consistent and truth-non-empty [in particular, classical]), in which case $\theta^{\mathcal{A}} = \Delta_A$, for $\chi^{\mathcal{A}}$ is injective, and so $\mathcal{A}$ is simple. Then, by Theorems 3.20 and 4.1(vi)⇒(i) with $\mathsf{S} = \{\mathcal{A}\}$, the logic of $\mathcal{A}$ is self-extensional, its intrinsic variety being generated by $\mathfrak{A}$. Thus, by the self-extensionality of inferentially inconsistent logics, *any two-valued logic is self-extensional.* □

Nevertheless, the procedure involved is simplified much under hereditary simplicity as well as either implicativity or both conjunctivity and disjunctivity of finitely many finite defining matrices upon the basis of the item (v) of Theorem 4.1.

4.1. **Self-extensionality of conjunctive disjunctive logics versus distributive lattices.**

Remark 4.3. Let C be a $\bar{\wedge}$-conjunctive or/and $\underline{\vee}$-disjunctive Σ-logic and $\phi \approx \psi$ a semi-lattice/"distributive lattice" identity for $\bar{\wedge}$ or/and $\underline{\vee}$. Then, $\phi \equiv^{\omega}_{C} \psi$. □

Theorem 4.4. *Let C be a $\diamond$-conjunctive/-disjunctive Σ-logic (defined by a class M of simple Σ-matrices) and $i = (0/1)$ (as well as $\mathsf{K} \triangleq \pi_0[\mathsf{M}]$). Then, C is self-extensional iff the following hold:*

(i) *each element of $\mathrm{IV}(C)(= \mathbf{V}(\mathsf{K}))$ is a $\diamond$-semi-lattice;*
(ii) *for all $\bar{\varphi} \in (\mathrm{Fm}^{\omega}_{\Sigma})^2$, $(\varphi_1 \in C(\varphi_0))$⇔|⇒$(\mathrm{IV}(C) \models (\varphi_i \approx (\varphi_0 \diamond \varphi_1))$.*

Proof. The "if" part is by Theorem 4.1(ii)⇒(i) and semi-lattice identities (more specifically, the commutativity one) for $\diamond$. Conversely, if C is self-extensional, then, by Theorem 4.1(i)⇒(iii), we have $\equiv^{\omega}_{C} = \theta^{\omega}_{\mathrm{IV}(C)}$, in which case, since C is $\diamond$-conjunctive/-disjunctive, (i) is by Remark 4.3 (and Theorem 3.20), while, for all $\bar{\varphi} \in (\mathrm{Fm}^{\omega}_{\Sigma})^2$, $(\varphi_1 \in C(\varphi_0)) \Leftrightarrow (\varphi_i \equiv^{\omega}_{C} (\varphi_0 \diamond \varphi_1))$, so (ii) holds. □

Lemma 4.5. *A [truth-non-empty $\bar{\wedge}$-conjunctive] Σ-matrix $\mathcal{A}$ is a $(2 \setminus 1)$-model of a [finitary $\bar{\wedge}$-conjunctive] Σ-logic C if[f] $\mathcal{A} \in \mathrm{Mod}(C)$ (cf. Definition 2.7).*

Proof. The "if" part is trivial. [Conversely, assume $\mathcal{A} \in \mathrm{Mod}_{2\setminus 1}(C)$. Consider any $\varphi \in C(\varnothing)$ and any $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$, in which case $V \triangleq \mathrm{Var}(\varphi) \in \wp_{\omega}(\mathrm{Var}_{\omega})$, and so $(\mathrm{Var}_{\omega} \setminus V) \neq \varnothing$, for, otherwise, we would have $V = \mathrm{Var}_{\omega}$, and so would get $\omega = |\mathrm{Var}_{\omega}| = |V| \in \omega$. Take any $v \in (\mathrm{Var}_{\omega} \setminus V)$ and any $a \in D^{\mathcal{A}} \neq \varnothing$. Let $g \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$ extend $(h{\restriction}(V \setminus \{v\})) \cup [v/a]$. Then, $\varphi \in C(v)$, $\{v\} \in \wp_{2\setminus 1}(\mathrm{Fm}^{\omega}_{\Sigma})$ and $g(v) = a \in D^{\mathcal{A}}$, in which case $h(\varphi) = g(\varphi) \in D^{\mathcal{A}}$, for $\mathcal{A} \in \mathrm{Mod}_{2\setminus 1}(C)$, and so $\mathcal{A} \in \mathrm{Mod}_2(C)$. By induction on any $n \in \omega$, let us prove that $\mathcal{A} \in \mathrm{Mod}_n(C)$. For consider any $X \in \wp_n(\mathrm{Fm}^{\omega}_{\Sigma})$, in which case $n \neq 0$. In case $|X| \in 2$, $X \in \wp_2(\mathrm{Fm}^{\omega}_{\Sigma})$, and so $C(X) \subseteq \mathrm{Cn}^{\omega}_{\mathcal{A}}(X)$, for $\mathcal{A} \in \mathrm{Mod}_2(C)$. Otherwise, $|X| \geqslant 2$, in which case there are some distinct $\phi, \psi \in X$, and so $Y \triangleq ((X \setminus \{\phi, \psi\}) \cup \{\phi \bar{\wedge} \psi\}) \in \wp_{n-1}(\mathrm{Fm}^{\omega}_{\Sigma})$. Then, by the induction hypothesis and the $\bar{\wedge}$-conjunctivity of both C and $\mathcal{A}$, $C(X) = C(Y) \subseteq \mathrm{Cn}^{\omega}_{\mathcal{A}}(Y) = \mathrm{Cn}^{\omega}_{\mathcal{A}}(X)$. So, $\mathcal{A} \in \mathrm{Mod}(C)$, as $\omega = (\bigcup \omega)$, and C is finitary.] □

Theorem 4.6. *Let C be a $\bar{\wedge}$-conjunctive [$\underline{\vee}$-disjunctive] Σ-logic and $\mathsf{V} \triangleq \mathrm{IV}(C)$ (as well as M a class of simple Σ-matrices defining C, and $\mathsf{K} \triangleq \pi_0[\mathsf{M}]$). {Suppose C is finitary (in particular, both M and all elements of it are finite).} Then, (i)⇔(ii){⇒}(iii)(⇒(iv))⇒(i), where:*

(i) *C is self-extensional;*
(ii) *for all $\phi, \psi \in \mathrm{Fm}^{\omega}_{\Sigma}$, it holds that $(\psi \in C(\phi)) \Leftrightarrow | \Rightarrow (\mathsf{V} \models (\phi \approx (\phi \bar{\wedge} \psi)))$, while every element of V is a $\bar{\wedge}$-semi-lattice [resp., distributive $(\bar{\wedge}, \underline{\vee})$-lattice];*

(iii) *every truth-non-empty* $\overline{\wedge}$*-conjunctive [consistent* $\veebar$*- disjunctive]* Σ*-matrix with underlying algebra in* V *is a model of* C*, while every element of* V *is a* $\overline{\wedge}$*-semi-lattice [resp., distributive* $(\overline{\wedge},\veebar)$*-lattice];*

(iv) *any truth-non-empty* $\overline{\wedge}$*-conjunctive [consistent* $\veebar$*- disjunctive]* Σ*-matrix with underlying algebra in* K *is a model of* C*, while every element of* K *is a* $\overline{\wedge}$*-semi-lattice [resp., distributive* $(\overline{\wedge},\veebar)$*-lattice].*

{*(In particular, (i–iv) are equivalent.)*}

Proof. First, (i)⇔(ii) is by Remark 4.3 and Theorem 4.4 with $i = 0$ and $\diamond = \overline{\wedge}$. {Next, (ii)⇒(iii) is by Lemma 4.5.} (Further, (iv) is a particular case of (iii), in view of Theorem 3.20.) Finally, assume (iii) (resp., (iv)) holds. Let S be the class of all truth-non-empty $\overline{\wedge}$-conjunctive [consistent $\veebar$- disjunctive] Σ-matrices with underlying algebra in V (resp., in K). Consider any $\mathcal{A} \in \mathsf{S}$ and any $\bar{a} \in (A^2 \setminus \Delta_A)$, in which case, by the semi-lattice identities ⟨more specifically, the commutativity one⟩ for $\overline{\wedge}$, $a_i \neq (a_i \mathbin{\overline{\wedge}^{\mathfrak{A}}} a_{1-i})$, for some $i \in 2$, and so $\mathcal{B} \triangleq \langle \mathfrak{A}, \{b \in A \mid a_i = (a_i \mathbin{\overline{\wedge}^{\mathfrak{A}}} b)\}\rangle \in \mathsf{S}$ [resp., by the Prime Ideal Theorem, there is some $\mathcal{B} \in \mathsf{S}$] such that $\mathfrak{B} = \mathfrak{A}$ and $a_i \in D^{\mathcal{B}} \not\ni a_{1-i}$. In this way, (i) is by Theorem(s) 4.1(vi)⇒(i) (and 3.20). □

Theorem 4.7. *Let* M *be a [finite] class of [finite hereditarily] simple [*$\overline{\wedge}$*-conjunctive* $\veebar$*-disjunctive]* Σ*-matrices,* $\mathsf{K} \triangleq \pi_0[\mathsf{M}]$ *and* C *the logic of* M*. Then,* C *is self-extensional if[f], for each* $\mathfrak{A} \in \mathsf{K}$ *and all distinct* $a, b \in A$*, there are some* $\mathcal{B} \in \mathsf{M}$ *and some* $h \in \hom(\mathfrak{A},\mathfrak{B})$ *such that* $\chi^{\mathcal{B}}(h(a)) \neq \chi^{\mathcal{B}}(h(b))$*.*

Proof. The "if" part is by Theorem 4.1(v)⇒(i) with $\mathsf{C} = \mathsf{K}$. [Conversely, assume C is self-extensional. Consider any $\mathfrak{A} \in \mathsf{K}$ and any $\bar{a} \in (A^2 \setminus \Delta_A)$. Then, by Theorem 4.6(i)⇒(iv), $\mathfrak{A}$ is a distributive $(\overline{\wedge},\veebar)$-lattice, in which case, by the commutativity identity for $\overline{\wedge}$, $a_i \neq (a_i \mathbin{\overline{\wedge}^{\mathfrak{A}}} a_{1-i})$, for some $i \in 2$, and so, by the Prime Ideal Theorem, there is some $\overline{\wedge}$-conjunctive $\veebar$-disjunctive Σ-matrix $\mathcal{D}$ with $\mathfrak{D} = \mathfrak{A}$ such that $a_i \in D^{\mathcal{D}} \not\ni a_{1-i}$, in which case $\mathcal{D}$ is both consistent and truth-non-empty, and so is a model of C. Hence, by Theorem 3.9 and Remark 2.6(ii), there are some $\mathcal{B} \in \mathsf{M}$ and some strict $h \in \hom(\mathcal{D},\mathcal{B}) \subseteq \hom(\mathfrak{A},\mathfrak{B})$, in which case $h(a_i) \in D^{\mathcal{B}} \not\ni h(a_{1-i})$, so $\chi^{\mathcal{B}}(h(a_i)) = 1 \neq 0 = \chi^{\mathcal{B}}(h(a_{1-i}))$.] □

4.2. **Self-extensionality of implicative logics versus implicative intrinsic semi-lattices.** A Σ-algebra $\mathfrak{A}$ is called an $\sqsupset$*-implicative intrinsic semi-lattice [with bound (a)]*, provided it is a $\uplus_{\sqsupset}$-semi-lattice [with bound (a)] and satisfies:

$$(x_0 \sqsupset x_0) \approx (x_1 \sqsupset x_1), \tag{4.1}$$

$$((x_0 \sqsupset x_0) \sqsupset x_1) \approx x_1, \tag{4.2}$$

in which case it is that with bound $a \sqsupset^{\mathfrak{A}} a$, for any $a \in A$.

Remark 4.8. Let C be a [self-extensional] Σ-logic and $\phi, \psi \in C(\varnothing)$, in which case $\phi \equiv^{\omega}_{C} \psi$ [and so $\mathrm{IV}(C) \models (\phi \approx \psi)$]. □

Theorem 4.9. *Let* M *be an* $\sqsupset$*-implicative* Σ*-logic* C *(defined by a class* M *of simple* Σ*-matrices and* $\mathsf{K} \triangleq \pi_0[\mathsf{M}]$*). Then,* C *is self-extensional iff, for all* $\phi, \psi \in \mathrm{Fm}^{\omega}_{\Sigma}$*, it holds that* $(\psi \in C(\phi))$⇔|⇒$(\mathrm{IV}(C) \models (\psi \approx (\phi \uplus_{\sqsupset} \psi)))$*, while each element of* $\mathrm{IV}(C)(= \mathbf{V}(\mathsf{K}))$ *is an* $\sqsupset$*-implicative intrinsic semi-lattice.*

Proof. First, by (2.5), Remark 4.8 and the strucuruality of C, $(4.1) \in \equiv^{\omega}_{C}$. Likewise, by (2.5), (2.6) and (2.7), $(4.2) \in \equiv^{\omega}_{C}$. Then, Theorems 3.5(ii) and 4.4 with $i = 1$ and $\diamond = \uplus_{\sqsupset}$ complete the argument. □

Lemma 4.10. *Let* C' *be a finitary* Σ*-logic and* C'' *a* 1*-extension of* C' *(cf. Definition 2.3). Suppose* C' *has DT with respect to* $\sqsupset$*, while* (2.7) *is satisfied in* C''*. Then,* C'' *is an extension of* C'*.*

Proof. By induction on any $n \in \omega$, we prove that C'' is an n-extension of C'. For consider any $X \in \wp_n(\mathrm{Fm}^\omega_\Sigma)$, in which case $n \neq 0$, and any $\psi \in C'(X)$. Then, in case $X = \varnothing$, we have $X \in \wp_1(\mathrm{Fm}^\omega_\Sigma)$, and so $\psi \in C'(X) \subseteq C''(X)$, for C'' is a 1-extension of C'. Otherwise, take any $\phi \in X$, in which case $Y \triangleq (X \setminus \{\phi\}) \in \wp_{n-1}(\mathrm{Fm}^\omega_\Sigma)$, and so, by DT with respect to $\sqsupset$, that C' has, and the induction hypothesis, we have $(\phi \sqsupset \psi) \in C'(Y) \subseteq C''(Y)$. Therefore, by (2.7)$[x_0/\phi, x_1/\psi]$ satisfied in C'', in view of its structurality, we eventually get $\psi \in C''(Y \cup \{\phi\}) = C''(X)$. Hence, as $\omega = (\bigcup \omega)$, we conclude that C'' is an extension of C', for this is finitary. □

Theorem 4.11. *Let M be a [finite] class of [finite hereditarily] simple [$\sqsupset$-implicative] Σ-matrices, $\mathsf{K} \triangleq \pi_0[\mathsf{M}]$ and C the logic of M. Then, C is self-extensional if[f], for each $\mathfrak{A} \in \mathsf{K}$ and all distinct $a, b \in A$, there are some $\mathcal{B} \in \mathsf{M}$ and some $h \in \hom(\mathfrak{A}, \mathfrak{B})$ such that $\chi^{\mathcal{B}}(h(a)) \neq \chi^{\mathcal{B}}(h(b))$.*

Proof. The "if" part is by Theorem 4.1(v)⇒(i) with $\mathsf{C} = \mathsf{K}$. [Conversely, assume C is self-extensional. Consider any $\mathfrak{A} \in \mathsf{K}$ and any $\bar{a} \in (A^2 \setminus \Delta_A)$. Then, by Theorem 4.9, $\mathfrak{A} \in \mathrm{IV}(C)$ is an $\sqsupset$-implicative intrinsic semi-lattice, in which case, by the commutativity identity for $\uplus_\sqsupset$, $a_{1-i} \neq (a_i \uplus^{\mathfrak{A}}_\sqsupset a_{1-i})$, for some $i \in 2$. Let $n \triangleq |A| \in (\omega \setminus 1)$. Take any bijective $\bar{c} : n \to A$. Let $g \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{A})$ extend $[x_j/c_j; x_k/c_0]_{j \in n; k \in (\omega \setminus n)}$, in which case $A = (\mathrm{img}\,\bar{c}) \subseteq (\mathrm{img}\,g) \subseteq A$, and so there is some $\bar{\varphi} \in (\mathrm{Fm}^\omega_\Sigma)^2$ such that $g(\bar{\varphi}) = \bar{a}$. Then, by (2.14), $S \triangleq g^{-1}[\mathrm{Fg}^{\mathfrak{A}}_C(\varnothing)] \in \mathrm{Fi}_C(\mathfrak{Fm}^\omega_\Sigma)$. Let us prove, by contradiction, that $\varphi_{1-i} \notin T \triangleq C(S \cup \{\varphi_i\})$. For suppose $\varphi_{1-i} \in T$, in which case, by DT, $(\varphi_i \sqsupset \varphi_{1-i}) \in C(S)$, and so $(\varphi_i \sqsupset \varphi_{1-i}) = \sigma(\varphi_i \sqsupset \varphi_{1-i}) \in S$, for $\sigma[S] = S \subseteq S$, where σ is the diagonal Σ-substitution. Then, $(a_i \sqsupset^{\mathfrak{A}} a_{1-i}) \in \mathrm{Fg}^{\mathfrak{A}}_C(\varnothing)$. Clearly, by (2.5), $F \triangleq \{a_i \sqsupset^{\mathfrak{A}} a_i\} \subseteq \mathrm{Fg}^{\mathfrak{A}}_C(\varnothing)$. Conversely, consider any $\phi \in C(\varnothing)$ and any $e \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{A})$, in which case, by the structurality of C, $\sigma'(\phi) \in C(\varnothing)$, where σ' is the Σ-substitution extending $[x_l/x_{l+1}]_{l \in \omega}$, and so, by (2.5) and Remark 4.8, $e(\phi) = e'(\sigma'(\phi)) = e'(x_0 \sqsupset x_0) = (a_i \sqsupset^{\mathfrak{A}} a_i) \in F$, where $e' \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{A})$ extends $[x_0/a_i; x_{m+1}/e(x_m)]_{m \in \omega}$ (in particular, $\mathcal{D} \triangleq \langle \mathfrak{A}, F \rangle \in \mathrm{Mod}_1(C)$; cf. Definition 2.7). And what is more, by (4.2), (2.7) is true in $\mathcal{D}$, in which case, by Lemma 4.10, $F \in \mathrm{Fi}_C(\mathfrak{A})$, and so $\mathrm{Fg}^{\mathfrak{A}}_C(\varnothing) \subseteq F$ (in particular, $\mathrm{Fg}^{\mathfrak{A}}_C(\varnothing) = F$). In this way, $(a_i \sqsupset^{\mathfrak{A}} a_{1-i}) = (a_i \sqsupset^{\mathfrak{A}} a_i)$, in which case, by (4.2), $(a_i \uplus^{\mathfrak{A}}_\sqsupset a_{1-i}) = ((a_i \sqsupset^{\mathfrak{A}} a_i) \sqsupset^{\mathfrak{A}} a_{1-i}) = a_{1-i}$, and so this contradiction shows that $\varphi_{1-i} \notin T$. Hence, there are some $\mathcal{B} \in \mathsf{M}$ and some $f \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{B})$ such that $(S \cup \{\varphi_i\}) \subseteq f^{-1}[D^{\mathcal{B}}] \not\ni \varphi_{1-i}$. Consider any $\bar{\psi} \in (\ker g)$. Let $\mathcal{E} \triangleq \langle \mathfrak{A}, \mathrm{Fg}^{\mathfrak{A}}_C(\varnothing) \rangle \in \mathrm{Mod}(C)$, $\theta \triangleq \partial(\mathcal{E}) \in \mathrm{Con}(\mathfrak{A})$ and $g' \triangleq (g \circ \nu_\theta) \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{A}/\theta)$, in which case $\bar{\psi} \in (\ker g')$, while $\nu_\theta \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{E}, \mathcal{E}/\theta)$, and so $S = g'^{-1}[D^{\mathcal{E}/\theta}]$. Then, by (2.14), Remark 2.6(ii,iv), Lemma 3.7 and Theorem 3.2, there is an axiomatic canonical equality determinant $\Xi \subseteq \mathrm{Fm}^2_\Sigma$ for $(\mathsf{M} \cup (\mathbf{ISPS}\mathsf{M})) \supseteq \{\mathcal{B}, \mathcal{E}/\theta\}$, in which case $(\Xi[x_l/\psi_l]_{l \in 2}) \subseteq S \subseteq f^{-1}[D^{\mathcal{B}}]$, and so $\bar{\psi} \in (\ker f)$. Thus, $(\ker g) \subseteq (\ker f)$, in which case, by the Homomorphism Theorem, $h \triangleq (g^{-1} \circ f) \in \hom(\mathfrak{A}, \mathfrak{B})$, and so $h(a_i) = f(\varphi_i) \in D^{\mathcal{B}} \not\ni f(\varphi_{1-i}) = h(a_{1-i})$, as required. □

4.3. **Self-extensionality of uniform finitely-valued logics versus truth discriminators.** A *truth discriminator for/of* a Σ-matrix $\mathcal{A}$ is any $\bar{h} : \mathrm{img}[\theta^{\mathcal{A}} \setminus \Delta_A] \to \hom(\mathfrak{A}, \mathfrak{A})$ such that, for every $\{a, b\} \in (\mathrm{dom}\,\bar{h})$, $\langle a, b \rangle \notin \ker(h_{\{a,b\}} \circ \chi^{\mathcal{A}})$. Then, since $\Delta_A \in \hom(\mathfrak{A}, \mathfrak{A})$, by Theorems 4.7 and 4.11, we have:

Corollary 4.12. *Let $\mathcal{A}$ be a [finite hereditarily] simple [either implicative or both conjunctive and disjunctive] Σ-matrix and C the logic of $\mathcal{A}$. Then, C is self-extensional if[f] $\mathcal{A}$ has a truth discriminator.*

The effective procedure of verifying the self-extensionality of the logic of an n-valued, where $n \in (\omega \setminus 1)$, hereditarily simple either implicative or both conjunctive and disjunctive Σ-matrix resulted from Corollary 4.12 has the computational complexity n^{n+2} that is quite acceptable for (3|4)-valued logics. And what is more, it provides a quite useful heuristic tool of doing it, manual applications of which (suppressing the factor n^{n+2} at all) are presented below. First, we have:

Corollary 4.13. *The logic of any no-less-than-three-valued hereditarily simple either implicative or both conjunctive and disjunctive Σ-matrix $\mathcal{A}$ without non-diagonal non-singular endomorphism of $\mathfrak{A}$ (cf. pp. 2,3) is not self-extensional.*

Proof. By contradiction. For suppose the logic of $\mathcal{A}$ is self-extensional, in which case, as $|A| \geqslant 3 \nleqslant 2$, $\chi^{\mathcal{A}}$ is not injective, and so there are some distinct $a, b \in A$ such that $\chi^{\mathcal{A}}(a) = \chi^{\mathcal{A}}(b)$. Then, by Corollary 4.12, there is some $h \in \hom(\mathfrak{A}, \mathfrak{A})$ such that $\chi^{\mathcal{A}}(h(a)) \neq \chi^{\mathcal{A}}(h(b))$, in which case $h(a) \neq h(b)$, and so h is not singular (in particular, diagonal). Hence, $\chi^{\mathcal{A}}(a) = \chi^{\mathcal{A}}(h(a)) \neq \chi^{\mathcal{A}}(h(b)) = \chi^{\mathcal{A}}(b) = \chi^{\mathcal{A}}(a)$. $\square$

4.3.1. *Self-extensionality versus equational implications and unitary equality determinants.* According to [19], given any $m, n \in \omega$, a *[finitary] (Σ-)equational* $\vdash^m_n$-*{sequent }definition for/of* a Σ-matrix $\mathcal{A}$ is any $\mho \in \wp_{[\omega]}(\mathrm{Eq}^{m+n}_{\Sigma})$ such that, for all $\bar{a} \in A^m$ and all $\bar{b} \in A^n$, it holds that $(((\mathrm{img}\, a) \subseteq D^{\mathcal{A}}) \Rightarrow (((\mathrm{img}\, b) \cap D^{\mathcal{A}}) \neq \varnothing)) \Leftrightarrow (\mathfrak{A} \models (\bigwedge \mho)[x_i/a_i; x_{m+j}/b_j]_{i \in m; j \in n})$. Equational $\vdash^{0/1}_1$-definitions are also referred to as *equational "truth [predicate] definitions"/implications* /(cf. [21]). Some kinds of equational sequent definitions are equivalent for implicative matrices, in view of:

Remark 4.14. Given a(n $\sqsupset$-implicative) Σ-matrix $\mathcal{A}$, (i) holds (as well as (ii–iv) do so), where:

(i) given a [finitary] equational $\vdash^2_2$-definition $\mho$ for $\mathcal{A}$, $\mho[x_{(2 \cdot i)+j}/x_i]_{i,j \in 2}$ is a [finitary] equational implication for $\mathcal{A}$ (cf. Theorems 10 and 12(ii)$\Rightarrow$(iii) of [19]);
(ii) given any [finitary] equational implication $\mho$ for $\mathcal{A}$, $\mho[x_0/(x_0 \sqsupset x_0), x_1/x_0]$ is a [finitary] equational truth definition for $\mathcal{A}$;
(iii) given any [finitary] equational truth definition $\mho$ for $\mathcal{A}$, $\mho[x_0/(x_0 \sqsupset (x_1 \sqsupset (x_2 \uplus_{\sqsupset} x_3)))]$ is a [finitary] equational $\vdash^2_2$-definition for $\mathcal{A}$;
(iv) in case $\mathcal{A}$ is truth-singular, $\{x_0 \approx (x_0 \sqsupset x_0)\}$ is a finitary equational truth definition for it. $\square$

In this way, taking Theorems 10, 12(i)$\Leftrightarrow$(ii) and 13 of [19] as well as Remark 4.14 into account, an either implicative or both conjunctive and disjunctive no-less-than-two-valued finite Σ-matrix $\mathcal{M}$ with unitary equality determinant has a finitary equational implication iff the multi-conclusion two-side sequent calculus $\widetilde{\mathcal{S}}^{(k,l)}_{\mathcal{M},\mathcal{T}}$ (cf. [18] as well as the paragraph -2 on p. 294 of [19] for more detail) is algebraizable (in the sense of [17, 16]). Then, by Lemma 9 and Theorem 10 of [19] as well as Corollary 4.13, we immediately get:

Corollary 4.15. *The logic of any no-less-than-tree-valued either implicative or both conjunctive and disjunctive Σ-matrix with unitary equality determinant and equational implication is not self-extensional.*

As a first generic application of the "implicative" parts of Remark 4.14 and Corollary 4.15, we have:

Corollary 4.16. *The logic of any no-less-than-tree-valued implicative truth-singular Σ-matrix with unitary equality determinant is not self-extensional.*

Example 4.17 (Łukasiewicz' finitely-valued logics; cf. [8])**.** Let $n \in (\omega \setminus 2)$, $\Sigma \triangleq (\Sigma_+ \cup \{\sim, \supset\})$ with binary $\supset$ (implication) and unary $\sim$ (negation) and $\mathcal{A}$ the Σ-matrix with $(\mathfrak{A} {\restriction} \Sigma_+) \triangleq \mathfrak{D}_n$ (cf. Subparagraph 2.2.1.2.1), $D^{\mathcal{A}} \triangleq \{1\}$, $\sim^{\mathfrak{A}} \triangleq (1 - a)$ and $(a \supset^{\mathfrak{A}} b) \triangleq \min(1, 1 - a + b)$, for all $a, b \in A$, in which case $\mathcal{A}$ is both consistent, truth-non-empty, $\wedge$-conjunctive and $\vee$-disjunctive as well as has both an equational implication, by Example 7 of [19], and a unitary equality determinant, by Example 3 of [18] (cf. Proposition 6.10 of [20] for a constructive proof of it). Hence, by Corollary 4.15, the logic of $\mathcal{A}$ is not self-extensional, unless $n = 2$. On the other hand, by induction on any $m \in (\omega \setminus 1)$, define the secondary unary connective $m \otimes x_0$ of Σ setting $((1[+m]) \otimes x_0) \triangleq ([\sim x_0 \supset (m \otimes]x_0)$, in which case $(m \otimes^{\mathfrak{A}} a) = \min(1, m \cdot a)$, for all $a \in A$, and so $\mathcal{A}$ is $((n-1) \otimes \sim x_0)$-negative (in particular, is implicative, for it is disjunctive; cf. Remark 2.8(i)(b)). In this way, the above negative result equally ensues from Example 3 of [18] and Corollary 4.16. □

This provides one of most representative applications of Corollary 4.16, another being discussed in Subparagraph 6.2.2.3.3 below (cf. Corollary 6.67 therein). On the other hand, in view of Theorem 10 and Lemma 8 of [19], Example(s) 4.2 [with $\Sigma = \Sigma_{+,01}$ and $\mathfrak{A} = \mathfrak{D}_{2,01}$; cf. Subparagraph 2.2.1.2.1] (and 4.17 with $n = 2$) as well as the self-extensionality of inferentially inconsistent {in particular, one-valued} logics, the stipulation "no-less-than-tree-valued" cannot be omitted in the formulation of Corollary 4.15 [4.13] (4.16).

Example 4.18. By Example 2 of [18], Remark 1 as well as Theorem 10 and Lemma 9 of [19] and Corollaries 4.13 and 4.15, arbitrary three-valued expansions of both the *logic of paradox* LP [13] and Kleene's three-valued logic KL_3 [6] are not self-extensional, for the matrix defining the former has the equational implication $(x_0 \wedge (x_1 \vee \sim x_1)) \approx (x_0 \wedge x_1)$, discovered in [15], while the matrix defining the latter has the same underlying algebra as that defining the former. Likewise, by "both Lemma 4.1 of [14] and Remark 4.14(i,iii)"/"Proposition 5.7 of [21]" as well as Corollary 4.15, arbitrary three-valued expansions of P^1/HZ [28]/[5] are not self-extensional, for their being defined by implicative/ matrices with equational "truth definition"/implication. □

Other generic applications of our universal elaboration presented in this section are discussed in Section 6.

5. Structural completions versus free models

Let M be a [finite] class of [finite] Σ-matrices, C the logic of M, $\mathsf{K} \triangleq \pi_0[\mathsf{M}]$ and α a [finite] Σ-rank. Then, for any $\mathcal{A} \in \mathsf{M}$ and any $h \in \hom(\mathrm{Fm}^{\alpha}_{\Sigma}, \mathfrak{A})$, $h \in \hom_{\mathrm{S}}(\mathcal{B}, \mathcal{A})$, where $\mathcal{B} \triangleq \langle \mathfrak{Fm}^{\alpha}_{\Sigma}, h^{-1}[D^{\mathcal{A}}]\rangle$, in which case, by Remark 2.6, we have $\theta^{\alpha}_{\mathsf{K}} \subseteq (\ker h) = h^{-1}[\Delta_A] \subseteq h^{-1}[\theta^{\mathcal{A}}] = \theta^{\mathcal{B}}$, and so $\theta^{\alpha}_{\mathsf{K}} \subseteq \theta^{\mathcal{D}}$, where $\mathcal{D} \triangleq \langle \mathfrak{Fm}^{\alpha}_{\Sigma}, \mathrm{Cn}^{\alpha}_{\mathsf{M}}(\varnothing)\rangle \in \mathrm{Mod}(C)$, in view of the structurality of C. Thus, $\theta^{\alpha}_{\mathsf{K}} \in \mathrm{Con}(\mathcal{D})$, in which case, by (2.14), $\mathcal{F}^{\alpha}_{\mathsf{M}} \triangleq (\mathcal{D}/\theta^{\alpha}_{\mathsf{K}}) \in \mathrm{Mod}(C)$, while $\mathfrak{F}^{\alpha}_{\mathsf{M}} = \mathfrak{F}^{\alpha}_{\mathsf{K}}$ [in particular, $\mathcal{F}^{\alpha}_{\mathsf{M}}$ is finite], whereas $I[= I^{\alpha}_{\mathsf{M}}] \triangleq ((\mathcal{B}^{\alpha}_{\mathsf{M}}[\cap\varnothing])[\cup\{\langle \mathcal{A}, f\rangle \mid \mathcal{A} \in \mathsf{M}, f : \mathrm{Var}_{\alpha} \to A\}])$ is a [finite] set [more precisely, $|I^{\alpha}_{\mathsf{M}}| \leqslant (\sum_{\mathcal{A}\in\mathsf{M}} \alpha^{|A|})$], and so choosing [resp., setting], for each $i \in I$, such $\mathcal{A}_i[\triangleq \pi_0(i)] \in \mathsf{M}$ and $h_i[\triangleq \pi_1(i)] \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A}_i)$ that $h_i^{-1}[D^{\mathcal{A}_i}] = i[\in \mathcal{B}^{\alpha}_{\mathsf{M}}]$, respectively, and then setting $\mathcal{E}_i \triangleq (\mathcal{A}_i \restriction (\mathrm{img}\, h_i))$, being the submatrix of $\mathcal{A}_i$ generated by $h_i[\mathrm{Var}_{\alpha}][= (\mathrm{img}\, h_i)$ to be found effectively], we eventually conclude that $\theta \triangleq \theta^{\alpha}_{\mathsf{K}} = (\mathrm{Eq}^{\alpha}_{\Sigma} \cap \bigcap_{i\in I}(\ker h_i))$, $g : \mathrm{Fm}^{\alpha}_{\Sigma} \to (\prod_{i\in I} E_i), \varphi \mapsto \langle h_i(\varphi)\rangle_{i\in I}$ is a strict surjective homomorphism from $\mathcal{D}$ onto the subdirect product $\mathcal{G}^{\alpha}_{\mathsf{M}} \triangleq ((\prod_{i\in I} \mathcal{E}_i)\restriction(\mathrm{img}\, g))$ of $\langle \mathcal{E}_i\rangle_{i\in I}$, being the submatrix of $\prod_{i\in I} \mathcal{E}_i$ generated by $g[\mathrm{Var}_{\alpha}]$ [to be found effectively], $(\ker g) = \theta$, and thus, by the Homomorphism Theorem, $e \triangleq (\nu_{\theta}^{-1} \circ g)$ is an isomorphism from $\mathcal{F}^{\alpha}_{\mathsf{M}}$ onto $\mathcal{G}^{\alpha}_{\mathsf{M}}$.

Theorem 5.1. *Let Σ be a signature [with(out) nullary symbols], M a [finite] class of {denumerably-generated [more specifically, finite]} ⟨weakly $\veebar$-disjunctive⟩ Σ-matrices, C the logic of M, [$f \in \prod_{\mathcal{A}\in\mathsf{M}} \wp_{\omega(\setminus 1)}(A)$,] $\alpha \triangleq (\omega[\cap((1\cup) \bigcup_{\mathcal{A}\in\mathsf{M}} |f(\mathcal{A})|])$*

and $\mathcal{B}$ a submatrix of $\mathcal{G}^{\alpha}_{\mathsf{M}}$. Suppose every $\mathcal{A} \in \mathsf{M}$ is a surjectively homomorphic image of $\mathcal{B}$, unless $\mathcal{B} = \mathcal{G}^{\alpha}_{\mathsf{M}}$, [and is generated by $f(\mathcal{A})$]. Then, the structural completion of C is defined by $\mathcal{B}$. In particular, C is structurally complete iff, for each denumerably-generated $\{$non-proper$\}$ $\lceil$non-$\rceil$simple consistent submatrix $\mathcal{E}$ of any $\langle\veebar$-disjunctive$\rangle$ element of M, there are some [finite] $\langle$one-element$\rangle$ set $I[\in (\alpha^{|B|}+1)]$, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ and some its subdirect product in $\mathbf{H}^{-1}(\mathcal{E}\lceil/\eth(\mathcal{E})\rceil)$.

Proof. First, by (2.14), the logic C' of $\mathcal{G}^{\omega[/\alpha]}_{\mathsf{M}}$ is defined by $\mathcal{D}_{\omega[/\alpha]} \triangleq \langle \mathfrak{Fm}^{\omega[/\alpha]}_{\Sigma}, \mathrm{Cn}^{\omega[/\alpha]}_{\mathsf{M}}(\varnothing)\rangle \in \mathrm{Mod}(C)$, in view of the structurality of C [/and (2.13)], in which case it is an extension of C, and so $C(\varnothing) \subseteq C'(\varnothing)$. For proving the converse inclusion, consider the following complementary cases:

- $\alpha = \omega$.
  Then, applying the diagonal Σ-substitution, we get $C'(\varnothing) \subseteq D^{\mathcal{D}_\omega} = C(\varnothing)$.
- $\alpha \neq \omega$.
  Consider any $\mathcal{A} \in \mathsf{M}$, in which case it is generated by $f(\mathcal{A})$ of cardinality $\leqslant \alpha$, and so there is some surjective $h \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A})$. Then, $D^{\mathcal{D}_\alpha} = \mathrm{Cn}^{\alpha}_{\mathsf{M}}(\varnothing) \subseteq h^{-1}[D^{\mathcal{A}}]$, in which case $h \in \hom^{\mathrm{S}}(\mathcal{D}_\alpha, \mathcal{A})$, and so, by (2.15), $C'(\varnothing) \subseteq C(\varnothing)$.

Next, $\mathcal{D}_\omega$ is a model of any extension C'' of C' such that $C''(\varnothing) = C(\varnothing)$, in view of its structurality [and so is its submatrix $\mathcal{D}_\alpha$, in view of (2.13) and (2.14)], in which case C' is the structural completion of C. Further, by (2.14), $\mathcal{B}$ is a model of C'. Conversely, if $\mathcal{B} = \{\neq\}\mathcal{G}^{\alpha}_{\mathsf{M}}$, then $\{$each $\mathcal{A} \in \mathsf{M}$ is a surjective homomorphic image of $\mathcal{B}$, in which case, by (2.15)$\}$ $\mathrm{Cn}_{\mathcal{B}}(\varnothing) = C'(\varnothing)$, and so C', being structurally complete, is defined by $\mathcal{B}$. Finally, as $|\mathrm{Var}_\omega| = \omega$, any Σ-matrix is a model of a Σ-logic iff each denumerably-generated submatrix of it is so. In this way, (2.14) as well as Lemma 3.7 $\langle$resp., Remarks 2.6(ii,iv), 2.8(ii)(a,b) and Theorem 3.9$\rangle$ complete the argument. □

This provides [effective] algebraic criteria of admissibility of [finitary] rules in and structural completeness of [finitely-valued] logics [so implying the decidability of this problem]. [On the other hand, the computational complexity of resulting effective procedures may be to large to count them *practically* applicable. For instance, when M consists of a single (without loss of generality, simple; cf. (2.14) and Remark 2.6(iv)) consistent truth-non-empty (cf. Remarks 2.9 and 2.5) n-valued Σ-matrix, where $n \in (\omega \setminus 2)$, n is the upper limit of α, in which case n^n is the upper limit of $|I^{\alpha}_{\mathsf{M}}|$, and so the upper limit of $|B|$ is $n^{(n^n)}$. In particular, the procedure of verifying admissibility of finitary Σ-rules of rank $m \in \omega$ in C has the computational complexity $(n^{(n^n)})^m$, being relatively acceptable, only if either $n = 2$ and $m \leqslant 26$ or $n = 3$ and $m \leqslant 2$. Likewise, in case the unique element of M is disjunctive, the computational complexity of the procedure of verifying structural completeness of C is $(n^{(n^n)})^n$, being relatively acceptable, only if $n = 2$. Otherwise, the situation is even much worse (more, precisely, the computational complexity of the procedure of verifying structural completeness of C is $n^{((n^{(n^n)})^{(n^{(n^{(n^n)})})})}$, being absolutely unacceptable, even if $n = 2$, especially taking refusal of Windows calculator to compute even its degree even in advanced mode into account). These general evaluations make the quite effective algebraic criteria of structural completeness to be obtained in the next section and suppressing such hyper-combinatorial factors at all more than acute.]

6. Applications to no-more-than-four-valued logics

All along throughout this section, $([\wr =]\sim)/ \supset$ is supposed to be a primary unary/binary connective of Σ viewed as negation/implication [unless otherwise specified]. Let $\Sigma^{\langle\supset\rangle\{\bar{\varsigma}\}}_{\sim(+)[01]} \triangleq (\{\sim\}(\cup\Sigma_+)[\cup\Sigma_{01}]\langle\cup\{\supset\}\rangle\{\cup(\operatorname{img}\bar{\varsigma})\})$ [(cf. Subparagraph 2.2.1.2.1)] {where $\bar{\varsigma}$ is a finite sequence of primary connectives not belonging to $\Sigma^{\langle\supset\rangle}_{\sim(+)[01]}$}.

6.1. **Uniform four-valued expansions of Belnap's four-valued logic.** A *[bounded] De Morgan lattice* [17] is any $\Sigma_{\sim,+[01]}$-algebra, with [bounded] distributive lattice $\Sigma_{+[01]}$-reduct satisfying:

$$\sim\sim x_0 \approx x_0, \tag{6.1}$$

$$\sim(x_0 \vee x_1) \approx (\sim x_0 \wedge \sim x_1), \tag{6.2}$$

By $\mathfrak{DM}_{4[01]}$ we denote the non-Boolean diamond [bounded] De Morgan lattice with $(\mathfrak{DM}_{4[01]}\restriction\Sigma_{+[01]}) \triangleq \mathfrak{D}^2_{2[01]}$ and $\sim^{\mathfrak{DM}_{4[01]}}\langle i,j\rangle \triangleq \langle 1-j, 1-i\rangle$, for all $i,j \in 2$. In this connection, we use standard abbreviations going back to [2]:

$$\mathsf{t} \triangleq \langle 1,1\rangle, \qquad \mathsf{f} \triangleq \langle 0,0\rangle, \qquad \mathsf{b} \triangleq \langle 1,0\rangle, \qquad \mathsf{n} \triangleq \langle 0,1\rangle,$$

Here, it is supposed that $\Sigma \supseteq \Sigma_{\sim,+[01]}$ and $(\bar{\wedge}|\bar{\vee}) = (\wedge|\vee)$. Fix a Σ-matrix $\mathcal{A}$ with $(\mathfrak{A}\restriction\Sigma_{\sim,+[01]}) \triangleq \mathfrak{DM}_{4[01]}$ and $D^{\mathcal{A}} \triangleq (2^2 \cap \pi_0^{-1}[\{1\}])$. Then, $\mathcal{A}$ as well as its submatrices are both $\wedge$-conjunctive and $\vee$-disjunctive as well as both consistent and truth-non-empty (cf. Remark 2.8(ii)(a,b)), while $\{x_0, \sim x_0\}$ is a unitary equality determinant for them (cf. Example 2 of [18]), so they are hereditarily simple (cf. Lemma 3.1). Let C be the logic of $\mathcal{A}$. Then, since $\mathcal{DM}_{4[01]} \triangleq (\mathcal{A}\restriction\Sigma_{\sim,+[01]})$ defines [the bounded version/expansion of] Belnap's four-valued logic $B_{4[01]}$ [2] (cf. [17, 23, 22, 25]), C is a uniform four-valued expansion of $B_{4[01]}$. Conversely, according to Corollary 4.9 of [23], any uniform four-valued expansion of $B_{4[01]}$ is defined by a unique expansion of $\mathcal{DM}_{4[01]}$, in which case $\mathcal{A}$ is uniquely determined by C, and so is said to be *characteristic for/of C*. Moreover, by (2.14), Remark 2.6(ii) and Theorem 3.9, C is $\sim$-subclassical iff Δ_2 forms a subalgebra of $\mathfrak{A}$, in which case $\mathcal{A}\restriction 2$ is isomorphic to any $\sim$-classical model of C, and so defines a unique $\sim$-classical extension of C (cf. Theorem 4.20 of [23]), in its turn, denoted by C^{PC} and relatively axiomatized according to Corollary 6.3 below.

Lemma 6.1. *C is $\sqsupset$-implicative iff $\mathcal{A}$ is so.*

Proof. The "if" part is immediate. Conversely, assume C is $\sqsupset$-implicative, in which case, by Theorem 3.5, it is $\uplus_\sqsupset$-disjunctive, and so, by the $\vee$-disjunctivity of $\mathcal{A}$ (in particular, of C), we have $C(x_0 \vee x_1) = (C(x_0) \cap C(x_1)) = C(x_0 \uplus_\sqsupset x_1)$. Then, by (2.9), $C(\varnothing) = C((x_0 \sqsupset x_1) \uplus_\sqsupset x_0)) = C((x_0 \sqsupset x_1) \vee x_0))$, in which case the axiom $(x_0 \sqsupset x_1) \vee x_0$ is true in $\mathcal{A}$ as well as both (2.7) and (2.6), being satisfied in C, are so, and so, $\mathcal{A}$, being $\vee$-disjunctive, is $\sqsupset$-implicative. □

Given any $i \in 2$, put $DM_{3,-,i} \triangleq (2^2 \setminus \{\langle i, 1-i\rangle\})$. Then, we have the submatrix $\mathcal{A}_{3,i}$ generated by $DM_{3,-,i}$ with carrier (not) distinct from the generating set (in particular, when, e.g., $\Sigma = \Sigma_{\sim,+[,01]}$), taking (2.14) into account, the logic $C_{3,i}$ of which is a both $\vee$-disjunctive and $\wedge$-conjunctive {for its defining matrix is so} as well as inferentially consistent {for its defining matrix is both consistent and truth-non-empty} uniform no-more-than-four-valued extension of C (and a three-valued expansion of [the bounded version/expansion $LP_{01}|KL_{3,01}$ of] "the *logic of paradox*"|"Kleene's three-valued logic" $LP|KL_3$ [13]|[6], whenever $i = (0|1)$, for $\mathcal{DM}_{3,i[01]} \triangleq (\mathcal{A}_{3,i}\restriction\Sigma_{\sim,+[01]})$ defines $LP_{[01]}|KL_{3[01]}$), in which case it is $\sim$-paraconsistent|$(\vee,\sim)$-paracomplete, and so is not $\sim$-classical, in view of Remark 2.8(i)(c|d).

6.1.1. *Miscellaneous expansions.*

6.1.1.1. Classically-negative expansions. Next, C is referred to as a *(purely) classical⟨ly-negative⟩ {uniform four-valued} expansion of* $B_{4[,01]}$, provided $(\Sigma \subseteq)\Sigma^{\neg}_{\sim,+[,01]} \subseteq \Sigma$, where $\neg$ — classical negation — is unary, and $\neg^{\mathfrak{A}}\langle i,j\rangle \triangleq \langle 1-i, 1-j\rangle$, for all $i,j \in 2$, in which case (we set $\mathcal{DMB}_{4[,01]} \triangleq \mathcal{A}$, while) $\mathcal{A}$ is $\neg$-negative, and so, being $\vee$-disjunctive, is $\sqsupset^{\neg}_{\vee}$-implicative (in particular, C is so), in view of Remark 2.8(i)(a).

6.1.1.2. Bilattice expansions. Likewise, C is referred to as a *(purely) bilattice {uniform four-valued} expansion of* $B_{4[,01]}$, provided $(\Sigma \subseteq)\Sigma^{\sqcap,\sqcup}_{\sim,+[,01]} \subseteq \Sigma$, where $\sqcap$ and $\sqcup$ — *knowledge/information* conjunction and disjunction — are binary, and $(\langle i,j\rangle(\sqcap|\sqcup)^{\mathfrak{A}}\langle k,l\rangle) \triangleq \langle(\min|\max)(i,k),(\max|\min)(j,l)\rangle$, for all $i,j,k,l \in 2$.

6.1.1.3. Implicative expansions. Finally, C is referred to as a *(purely) ⟨canonically⟩ implicative {uniform four-valued} expansion of* $B_{4[,01]}$, provided $(\Sigma \subseteq)\Sigma^{\supset}_{\sim,+[,01]} \subseteq \Sigma$ and $(\langle i,j\rangle \supset^{\mathfrak{A}} \langle k,l\rangle) \triangleq \langle \max(1-i,k), \max(1-i,l)\rangle$, for all $i,j,k,l \in 2$, in which case $\mathcal{A}$ is $\supset$-implicative, and so is C.

6.1.2. *Structural completeness versus maximal paracompleteness, paraconsistency and consistency as well as inconsistency of resolutional extensions.*

Lemma 6.2. *The following are equivalent:*

(i) $D_{3,-,1}$ *does not form a subalgebra of* $\mathfrak{A}^2$*;*
(ii) $\mathcal{A}_{3,1} = \mathcal{A}$*;*
(iii) $\mathcal{A}_{3,1}$ *is* $\sim$*-paraconsistent;*
(iv) C^{R} *is not defined by* $\mathcal{A}_{3,1}$*;*
(v) *providing* C *does [not] have theorems,* C^{R} *is not [inferentially]* $(\vee,\sim)$*-paracomplete;*
(vi) *providing* C *does [not] have theorems,* $C^{\mathrm{R}} = C^{\mathrm{PC}}_{[+0]}$*, if* C *is* $\sim$*-subclassical (i.e.,* $\{\mathsf{f},\mathsf{t}\}$ *forms a subalgebra of* $\mathfrak{A}^2$*), and* C^{R} *is [inferentially] inconsistent, otherwise;*
(vii) C^{R} *is not an expansion of* KL_3*.*

Proof. First, (i)⇔(ii)⇔(iii) are immediate. Next, by Theorem 3.12, C^{R} is not $\sim$-paraconsistent, so (iii)⇒(iv) holds. Likewise, as $\mathcal{A}_{3,1}|KL_3$ is (inferentially) $(\vee,\sim)$-paracomplete, (iv|vii) is a particular case of (v). Furthermore, (iv/vii)⇒(i) is by (2.14) and Theorem 3.12. Further, (vi)⇒(v) is by [(2.12) and] the structurality of C^{R} as well as Remark 2.8(i)(d). Finally, (i)⇒(vi) is by Theorem 3.12 and Corollary 3.10(i)⇔(iv) [as well as Remark 2.5]. □

Then, by Corollary 2.9 of [23], Remarks 2.5, 2.8(i)(d), Lemma 6.2(iv)⇒(i)⇒(vi) and the $(\vee,\sim)$-paracompleteness of $\mathcal{A}_{3,1}$, we immediately have:

Corollary 6.3. *If* C *is* $\sim$*-classical (i.e.,* $\{\mathsf{f},\mathsf{t}\}$ *forms a subalgebra of* $\mathfrak{A}^2$*), then* C^{PC} *is relatively axiomatized by* $\{x_0 \vee \sim x_0, (3.2)\}$*.*

Lemma 6.4. *Let* $\mathcal{B} \in \mathbf{S}_*(\mathcal{A})$ *and* $h \in \hom^{\mathrm{S}}(\mathcal{A},\mathcal{B})$*. Then,* h *is diagonal. In particular,* $\mathcal{B} = \mathcal{A}$*.*

Proof. First, for every $a \in 2^2$, $(a \in \{\mathsf{b},\mathsf{n}\}) \Leftrightarrow (\sim^{\mathfrak{A}}a = a)$, in which case $\sim^{\mathfrak{A}}[\{\mathsf{b},\mathsf{n}\}] \subseteq \{\mathsf{b},\mathsf{n}\}$, and so, as $\mathsf{b} \in D^{\mathcal{A}}$, $h(\mathsf{b}) = \mathsf{b}$. Then, since $(\mathsf{f}|\mathsf{t}) = (\mathsf{b}(\wedge|\vee)^{\mathfrak{A}}\mathsf{n})$, $h(\mathsf{f}|\mathsf{t}) = (\mathsf{b}(\wedge|\vee)^{\mathfrak{A}}h(\mathsf{n}))$. Therefore, if $h(\mathsf{n})$ was not equal to n, then it would be equal to b, in which case $B = h[A]$ would be equal to $\{\mathsf{b}\}$, and so $\mathcal{B}$ would be inconsistent. Thus, $h(\mathsf{n}) = \mathsf{n}$, in which case $h(\mathsf{f}|\mathsf{t}) = (\mathsf{f}|\mathsf{t})$, and so h is diagonal, as required. □

Corollary 6.5. *Let* $\mathcal{B}$ *be a [consistent] (truth-non-empty) submatrix of* $\mathcal{A}$*. [Suppose* $\mathcal{A} \neq \mathcal{B}$ *has no inconsistent submatrix, whenever* $\mathcal{B}$ *is* $\sim$*-paraconsistent.] Then, the logic* C' *of* $\mathcal{D} = (\mathcal{A} \times \mathcal{B})$ *is an axiomatically-equivalent {and so (inferentially)* $(\vee,\sim)$*-paracomplete} [proper] extension of* C*.*

Proof. As $(\pi_0{\restriction}D) \in \hom^{\mathrm{S}}(\mathcal{D},\mathcal{A})$, the []-non-optional version is by (2.14,2.15) {for $\mathcal{A}$ is (both truth-non-empty and) $(\vee,\sim)$-paracomplete, and so is $\mathcal{D}$ (as $\mathcal{B}$ is truth-non-empty)}. [Finally, we prove that $C' \neq C$, by contradiction. For suppose $C' = C$, in which case $\mathcal{D}$ is $\sim$-paraconsistent, for $\mathcal{A}$ is so, and so there is some $a \in D^{\mathcal{D}}$ such that $\sim^{\mathfrak{D}} a \in D^{\mathcal{A}}$. Then, (2.10) is not true in $\mathcal{B}$ under $[x_0/\pi_1(a), x_1/b]$, where $b \in (B \setminus D^{\mathcal{B}}) \neq \varnothing$, for $\mathcal{B}$ is consistent, in which case $\mathcal{B}$ is $\sim$-paraconsistent, and so $\mathcal{A} \neq \mathcal{B}$ has no inconsistent submatrix. Moreover, $\mathcal{A}$ is a finite consistent $\vee$-disjunctive model of $C = C'$, being defined by $\mathcal{D}$, in which case $\mathcal{D} \in \mathrm{Mod}(C)$ is weakly $\vee$-disjunctive, for $\mathcal{A}$ is so, and so, by Theorem 3.9, there is some $h \in \hom_{\mathrm{S}}(\mathcal{A},\mathcal{D})$. Then, $g \triangleq (h \circ \pi_1)$ is a surjective homomorphism from $\mathcal{A}$ onto $\mathcal{E} \triangleq (\mathcal{B}{\restriction}(\mathrm{img}\, g)) \in \mathbf{S}(\mathcal{A}) = \mathbf{S}_*(\mathcal{A})$, in which case, by Lemma 6.4, $A = E \subseteq B \subseteq A$, and so $\mathcal{B} = \mathcal{A}$. This contradiction completes the argument.] □

Theorem 6.6. *The following are equivalent:*

(i) *C is maximally inferentially $(\vee,\sim)$-paracomplete;*

(ii) *the following hold:*

  (a) *$\mathcal{A}$ has no non-$\sim$-paraconsistent truth-non-empty consistent submatrix (viz., neither $\{\mathsf{f},\mathsf{t}\}$ nor $DM_{3,-,1}$ forms a subalgebra of $\mathfrak{A}$ [i.e., neither C is $\sim$-subclassical nor C^{R} is inferentially $(\vee,\sim)$-paracomplete {that is, C^{R} is inferentially inconsistent}; cf. Lemma 6.2(i)⇔(v){⇔(vi)}]);*

  (b) *$\mathcal{A}$ has no non-$(\vee,\sim)$-paracomplete truth-non-empty consistent submatrix (viz., in view of (a), $DM_{3,-,0}$ does not form a subalgebra of $\mathfrak{A}$ [i.e., C is maximally $\sim$-paraconsistent; cf. Theorem 4.31(i)⇔(iv) of* [23]*]), unless it has an inconsistent submatrix {i.e., $\{\mathsf{b}\}$ forms a subalgebra of $\mathfrak{A}$};*

(iii) *the following hold:*

  (a) *(ii)(a) holds;*

  (b) *$\mathcal{A}$ has no proper truth-non-empty consistent submatrix (viz., $\mathfrak{A}$ has no proper non-one-element subalgebra [i.e., C is maximally inferentially consistent; cf. Theorem 4.16 of* [23]*]), unless it has an inconsistent submatrix {i.e., $\{\mathsf{b}\}$ forms a subalgebra of $\mathfrak{A}$}.*

Proof. First, (i)⇒(iii) is by Corollary 6.5. Next, (ii) is a particular case of (iii), in view of the inferential $(\vee,\sim)$-paracompleteness of $\mathcal{A}$. Finally, assume (ii) holds. Consider any inferentially $(\vee,\sim)$-paracomplete extension C' of C, in which case $(x_0 \vee \sim x_0) \notin T \triangleq C'(x_1) \ni x_1$, while by the strucuturality of C', $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, T\rangle$ is a model of C' (in particular, of its sublogic C), and so is its finitely-generated $(\vee,\sim)$-paracomplete truth-non-empty submatrix $\mathcal{B} \triangleq \langle \mathfrak{Fm}^2_{\Sigma}, T \cap \mathrm{Fm}^2_{\Sigma}\rangle$, in view of (2.14). Then, by Lemma 3.7, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}^*_*(\mathcal{A})^I$ and some subdirect product $\mathcal{D} \in \mathbf{H}^{-1}(\mathcal{B}/\partial(\mathcal{B}))$ of it, in which case, by (2.14) and Remark 2.8(ii)(b), $\mathcal{D}$ is a $(\vee,\sim)$-paracomplete model of C', for $\mathcal{B}$ is so, and so there is some $a \in D$ such that $\{\mathsf{t},\mathsf{b},\mathsf{n}\}^I \ni b \triangleq (a \vee^{\mathfrak{D}} \sim^{\mathfrak{D}} a) \notin D^{\mathcal{A}}$ (in particular, $J \triangleq \{i \in I \mid \pi_i(a) = \mathsf{n}\} \neq \varnothing$, because, for any $c \in \{\mathsf{t},\mathsf{b},\mathsf{f}\}$, $(c \vee^{\mathfrak{A}} \sim^{\mathfrak{A}} c) \in D^{\mathcal{A}}$). Furthermore, by Claim 4.17 of [23], $f \triangleq (I \times \{\mathsf{f}\}) \in D \ni t \triangleq (I \times \{\mathsf{t}\})$. On the other hand, by (ii)(a), the submatrix $\mathcal{E}$ of $\mathcal{A}$ generated by $\{\mathsf{f},\mathsf{t}\}$, being both consistent and truth-non-empty, is $\sim$-paraconsistent, in which case $\mathsf{b} \in E$, and so there is some $\phi \in \mathrm{Fm}^2_{\Sigma}$ such that $\phi^{\mathfrak{A}}(\mathsf{f},\mathsf{t}) = \mathsf{b}$. Hence, $D \ni d \triangleq \phi^{\mathfrak{D}}(f,t) = (I \times \{\mathsf{b}\})$. Consider, the following complementary cases:

- $\{\mathsf{b}\}$ forms a subalgebra of $\mathfrak{A}$. Then, as $J \neq \varnothing$, $e : A \to A^I, c \mapsto ((J \times \{c\}) \cup ((I \setminus J) \times \{\mathsf{b}\}))$ is injective, in which case $D \ni d = e(\mathsf{b})$, and so $D \ni ((d \vee^{\mathfrak{D}} \sim^{\mathfrak{D}} b) \wedge^{\mathfrak{D}} b) = e(\mathsf{n})$. Therefore, since $\mathcal{A}$ is generated by $\{\mathsf{b},\mathsf{n}\}$, e is an embedding of $\mathcal{A}$ into $\mathcal{D}$.
- $\{\mathsf{b}\}$ does not form a subalgebra of $\mathfrak{A}$, in which case $\mathcal{A}$ has no inconsistent submatrix, and so no non-$(\vee,\sim)$-paracomplete truth-non-empty consistent

submatrix, in view of (ii)(b). Then, $\mathcal{E}$, being a both consistent and truth-non-empty submatrix of $\mathcal{A}$, is $(\vee,\sim)$-paracomplete, in which case $\mathsf{n} \in E$, and so there is some $\psi \in \mathrm{Fm}_\Sigma^2$ such that $\psi^{\mathfrak{A}}(\mathsf{f},\mathsf{t}) = \mathsf{n}$ (in particular, $D \ni \psi^{\mathfrak{D}}(f,t) = (I \times \{\mathsf{n}\})$). Therefore, since $I \supseteq J \neq \varnothing$, $\{\langle c, I \times \{c\}\rangle \mid c \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$.

Thus, anyway, $\mathcal{A}$ is embeddable into $\mathcal{D} \in \mathrm{Mod}(C')$, and so, by (2.14), $C' = C$. □

Since any Σ-logic with theorems is [in]consistent/$(\vee,\sim)$-paracomplete iff it is inferentially so, while any Σ-logic axiomatically-equivalent to a $(\vee,\sim)$-paracomplete one is $(\vee,\sim)$-paracomplete, whereas any $\sim$-paraconsistent Σ-matrix is truth-non-empty, by Remark 2.9, Corollaries 3.10(i)⇔(iv), 6.5 and Theorem 6.6, we eventually get:

Corollary 6.7. *The following are equivalent:*

(i) *C is structurally complete;*
(ii) *C is maximally $(\vee,\sim)$-paracomplete;*
(iii) *the following hold:*
  (a) *$\mathcal{A}$ has no non-$\sim$-paraconsistent consistent submatrix (viz., neither $\{\mathsf{f},\mathsf{t}\}$ nor $DM_{3,-,1}$ nor $\{\mathsf{n}\}$ forms a subalgebra of $\mathfrak{A}$ [i.e., neither C is $\sim$-subclassical nor C is purely-inferential nor C^{R} is $(\vee,\sim)$-paracomplete {that is, C^{R} is inconsistent}]);*
  (b) *$\mathcal{A}$ has no non-$(\vee,\sim)$-paracomplete consistent submatrix (viz., in view of (a), $DM_{3,-,0}$ does not form a subalgebra of $\mathfrak{A}$ [i.e., C is maximally $\sim$-paraconsistent]), unless it has an inconsistent submatrix {i.e., $\{\mathsf{b}\}$ forms a subalgebra of $\mathfrak{A}$};*
(iv) *the following hold:*
  (a) *(ii)(a) holds;*
  (b) *$\mathcal{A}$ has no proper consistent submatrix (viz., in view of (a), $\mathfrak{A}$ has no proper non-one-element subalgebra [i.e., C is maximally consistent; cf. Theorem 4.16 of* [23]*]), unless it has an inconsistent submatrix {i.e., $\{\mathsf{b}\}$ forms a subalgebra of $\mathfrak{A}$}.*

Theorem/Corollary 6.6/6.7 provides an effective algebraic criterion of maximal inferential/ $(\vee,\sim)$-paracompleteness /(viz., structural completeness) positively covering arbitrary bilattice uniform four-valued expansions of $B_{4/,01}$ (cf. Corollary 5.2 of [23]) as well as non-$\sim$-subclassical classically-negative uniform four-valued expansions of $B_{4[,01]}$ (cf. Corollary 5.1(i) therein) but negatively covering /both $\sim$-subclassical {in particular, purely} classically-negative uniform four-valued expansions of $B_{4[,01]}$ /"and purely-inferential {in particular, purely} bilattice uniform four-valued expansions of B_4" as well as (purely implicative uniform four-valued expansions of) $B_{4[,01]}$, because they are $\sim$-subclassical (cf. Corollary 5.3 therein).

6.1.3. *No-more-than-four-valued extensions and their self-extensionality.*

Lemma 6.8 (Key 4-valued Lemma)**.** *Let $\mathcal{B} \in \mathrm{Mod}(C)$. Then, the following hold:*

(i) *$\mathcal{B}$ is $\vee$-disjunctive, whenever it is either inconsistent or truth-empty or $\sim$-negative or [non-$\sim$-classically-defining or] no-more-than-$(4[-1])$-valued;*
(ii) *providing $\mathcal{B}$ is $\vee$-disjunctive [and (not) truth-empty| "either $\sim$-negative or $\sim$-classically-defining"|| $\sim$-paraconsistent/$(\vee,\sim)$-paracomplete], it is a strictly surjectively homomorphic counter-image of a submatrix of $\mathcal{A}$ with carrier in $\mathsf{S}_{4[+(-)\emptyset|\mathrm{C}\|\mathrm{P/PC}]} \triangleq ((\{\{01\},\Delta_2,2^2\} \cup \{D_{3,-,l} \mid l \in 2\})[\cap(\setminus)(\{\{01\}|\Delta_2\}\|\{2^2, DM_{3,-,0/1}\})])$.*

Proof. (i) By contradiction. For suppose $\mathcal{B}$ is not $\vee$-disjunctive. Then, taking Remarks 2.6(iv), 2.8(ii)(a,b) and (2.14) into account, without loss of generality, one can assume that $\mathcal{B}$ is simple, in which case, by Corollary 3.16 and Theorem 3.20, $\mathfrak{B}$ belongs to the variety generated by $\mathfrak{A}$, and so $\mathfrak{B}{\restriction}\Sigma_{\sim,+}$ is a De Morgan lattice (in particular, $\mathfrak{B}{\restriction}\Sigma_{+}$ is a distributive lattice), for $(\mathfrak{A}{\restriction}\Sigma_{\sim,+}) = \mathfrak{DM}_4$ is so. And what is more, $\mathcal{B} \in \mathrm{Mod}(C)$ is both $\wedge$-conjunctive and weakly $\vee$-disjunctive, for C is so. Hence, since $\mathcal{B}$ is not $\vee$-disjunctive, there are some $a, b \in (D \setminus D^{\mathcal{B}})$, in which case $c \triangleq (a \wedge^{\mathfrak{B}} b) \notin D^{\mathcal{B}}$, such that $d \triangleq (a \vee^{\mathfrak{B}} b) \in D^{\mathcal{B}}$ (in particular, $\mathcal{B}$ is both consistent and truth-non-empty), in which case $d \notin \{a, b, c\}$, and so $|\{a, b, c, d\}| = 4$. Therefore, if $\mathcal{B}$ was $\sim$-negative, then, by its $\wedge$-conjunctivity and (6.2), we would have $D^{\mathcal{B}} \not\ni \sim^{\mathfrak{B}} d = (\sim^{\mathfrak{B}} a \wedge^{\mathfrak{B}} \sim^{\mathfrak{B}} b) \in D^{\mathcal{B}}$. Thus, $|B| \leqslant 4$, in which case $B = \{a, b, c, d\}$ (in particular, $|B| = 4 \nleqslant 3$), and so $\mathcal{B}$ is not $\sim$-classically-defining. In this way, $\mathfrak{B}$ is a distributive $(\wedge, \vee)$-lattice with zero c and unit d, in which case, by (6.1) and (6.2), $\sim^{\mathfrak{B}}(c|d) = (d|c)$, and so, by (6.1), $\sim^{\mathfrak{B}}[\{a, b\}] \subseteq \{a, b\}$, for $(\{a, b\} \cap \{c, d\}) = \varnothing$. Consider the following cases:

- $\sim^{\mathfrak{B}} a = a$, in which case, by (6.1), $\sim^{\mathfrak{B}} b = b$, and so $e \triangleq \{\langle a, 10\rangle, \langle b, 01\rangle, \langle c, 00\rangle, \langle d, 11\rangle\}$ is an isomorphism from $\mathfrak{B}{\restriction}\Sigma_{\sim,+}$ onto $\mathfrak{DM}_4$. Furthermore, by Lemma 3.7, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$, some subdirect product $\mathcal{D}$ of it and some $h \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}, \mathcal{B})$, in which case, $(\{h \circ e\} \cup \{\pi_i{\restriction}D \mid i \in I\} \in \wp_\omega(\hom(\mathfrak{D}{\restriction}\Sigma_{\sim,+}, \mathfrak{DM}_4))$, while, by Remark 2.8(ii)(b), $\mathcal{D}$ is consistent (in particular, $I \neq \varnothing$), for $\mathcal{B}$ is so, whereas $(\bigcap_{i \in I} \ker(\pi_i{\restriction}D)) = \Delta_D \subseteq \ker(h \circ e) \neq D^2$, for $\mathrm{img}(h \circ e) = DM_4 = 2^2$ is not a singleton, and so, by Theorem 3.8 of [23], there is some $i \in I$ such that $\ker(\pi_i{\restriction}D) = \ker(h \circ e) = (\ker h)$, for e is injective. Therefore, by the Homomorphism Theorem, as $(\mathrm{img}\, h) = B$, $h^{-1} \circ \pi_i$ is an embedding of $\mathcal{B}$ into $\mathcal{A}$, in which case, by Remark 2.8(ii)(a), $\mathcal{B}$ is $\vee$-disjunctive.
- $\sim^{\mathfrak{B}} a \neq a$, in which case $\sim^{\mathfrak{B}} a = b$, and so, by (6.1), $\sim^{\mathfrak{B}} b = a$. Then, for each $e' \in B$, $(e'(\wedge|\vee)^{\mathfrak{B}} \sim^{\mathfrak{B}} e') = (c|d) \notin | \in D^{\mathcal{B}}$, in which case $\mathcal{B}$, being $\wedge$-conjunctive, satisfies both $x_0 \vee \sim x_0$ and (3.2). And what is more, $\{c, d\}$ forms a subalgebra of $\mathfrak{B}$, in which case, by (2.14), $\mathcal{B}{\restriction}\{c, d\}$ is a $\sim$-classical model of C, and so this is $\sim$-subclassical. Then, by Corollary 6.3, $\mathcal{B} \in \mathrm{Mod}(C^{\mathrm{PC}})$. Conversely, the logic of the consistent truth-non-empty model $\mathcal{B}$ of C is an inferentially consistent extension of C, in which case, by Theorem 4.21 of [23], $\mathcal{B}$ is $\sim$-classically-defining.

(ii) Since $\mathsf{S}_{4[+(-)\emptyset|\mathrm{C}\|\mathrm{P/PC}]}$ is the set of the carriers of all [those] elements of $\mathbf{S}_*(\mathcal{A})$ [which are (not) truth-empty|"either $\sim$-negative or $\sim$-classically-defining"$\|\sim$-paraconsistent/$(\vee, \sim)$-paracomplete], (2.14), Remarks 2.6 (ii), 2.8(ii)(a,b) and Theorem 3.9 complete the argument. □

By Theorem 4.10 of [23], (2.14), Examples 4.2, 4.18, Lemma 6.8 and the self-extensionality of inferentially inconsistent logics, we first have:

Theorem 6.9. *Let C' be a uniform no-more-than-four-valued proper (in particular, no-more-than-three-valued) extension of C. Then, the following are equivalent:*

(i) *C' is self-extensional;*
(ii) *C' is either inferentially inconsistent or $\sim$-classical;*
(iii) *for each $i \in 2$, if $DM_{3,-,i}$ forms a subalgebra of $\mathfrak{A}$, then $C' \neq C_{3,i}$.*

Since $\mathcal{DM}_4{\restriction}\{01\}$ is the only truth-empty submatrix of $\mathcal{DM}_4$, while $\{01\} \subseteq [\nsubseteq]DM_{3,-,1[-1]} \supseteq \Delta_2$, by Theorem 4.10 of [23], (2.14) and Lemma 6.8, we also get:

Theorem 6.10. *Let M be a class of no-more-than-four-valued models of C, C' the logic of M, $\mathsf{M}^{(*)[\sim/\not\sim]}_{\{0|1\}}$ the class of all (truth-non-empty) [$\sim$-classicaly-/non-$\sim$-classically-defining] $\{\sim$-paraconsistent$|(\vee, \sim)$-paracomplete$\}$ consistent elements of M and $\mathsf{M}_2 = (\mathsf{M}_0 \cap \mathsf{M}_1)$. Then, C' is defined by $\{\mathcal{A} \mid \mathsf{M}_2 \neq \varnothing\} \cup \{\mathcal{A}{\restriction}\{01\} \mid$*

$(\mathsf{M}\setminus\mathsf{M}^*) \neq \varnothing = \mathsf{M}_1^{*,\not\sim} = \mathsf{M}_2\}\cup\{\mathcal{A}{\restriction}\Delta_2 \mid (\bigcup_{i\in 2}\mathsf{M}_i^{*,\not\sim}) = \mathsf{M}_2 = \varnothing \neq \mathsf{M}^{\sim}\}\cup\bigcup_{i\in 2}\{\mathcal{A}_{3,i} \mid \mathsf{M}_i^{*,\not\sim} \neq \varnothing = \mathsf{M}_2\}$. *In particular, C is defined by any both $\sim$-paraconsistent and $(\vee,\sim)$-paracomplete no-more-than-four-valued model, so it has no both $\sim$-paraconsistent and $(\vee,\sim)$-paracomplete no-more-than-three-valued model.*

Taking (2.12), Theorems 6.9, 6.10, Remark 2.5 and Example 4.2 into account, it only remains to study the following no-more-than-four-valued extensions of C.
6.1.3.1. Double three-valued non-iniform and non-proper extensions. By (2.14), (providing, for each $i \in 2$, $DM_{3,i}$ forms a subalgebra of $\mathfrak{A}$) the logic C_3 of $\{\mathcal{A}_{3,i} \mid i \in 2\}$ is a both $\vee$-disjunctive and $\wedge$-disjunctive {for its defining matrices are so} as well as inferentially-consistent {for its defining matrices are both consistent and truth-non-empty} (proper) extension of C (for this is minimally four-valued; cf. Theorem 4.10 of [23]). Let $\mu : 2^2 \to 2^2, \langle i,j\rangle \mapsto \langle j,i\rangle$ be the *mirror/specular* function.

Theorem 6.11 (cf. [22, 25])**.** *It does hold that* (v)⇐(i)⇔(ii)⇔(iii)⇒(iv)[⇒(iii)], *where:*

- (i) *$C_{[3]}$ is self-extensional;*
- (ii) *[for each/some $i \in 2$] $(\mu[{\restriction}A_{3,i}]) \in \hom(\mathfrak{A}_{[3,i]},\mathfrak{A})$;*
- (iii) *$\mathfrak{A}$ has a(n injective) non-singular non-diagonal [partial] endomorphism — cf. pp. 2,3;*
- (iv) *$\mathcal{A}$ has no equational implication — cf. Subsubsection 4.3.1;*
- (v) *$C_{\langle 3\rangle}$ is $\sim$-subclassical.*

In particular, C_3 is self-extensional, whenever C is so.

Proof. First, the fact that (iv) is equivalent to the ()-non-optional []-optional version of (iii) is due to Theorems 10, 13 and 15 of [19], while the []-optional version of (iii) is a particular case of the []-non-optional one, whereas the ()-non-optional version of (iii) is a particular case of the ()-optional one, being, in its turn, a particular case of (ii), for μ is injective. Next, the fact that (i) implies the ()-non-optional version of (iii) is by Theorem 4.7, for $D^{\mathcal{A}}[\cap DM_{3,-,0}]$ has two distinct elements. [Furthermore, by the injectivity of μ and the fact that, for any $i \in 2$, $\mu[DM_{3,-,i}] = DM_{3,-,1-i}$, while $2 = \{i, 1-i\}$, the alternatives in (ii) are equivalent.] Further, assume (ii) holds. Consider [any $i \in 2$ and] any distinct $a, b \in A_{[3,i]}$, in which case there is some $j \in 2$ such that $\pi_j(a) \neq \pi_j(b)$, and so $\chi^{\mathcal{A}_{[3,k_j]}}(h_j(a)) \neq \chi^{\mathcal{A}_{[3,k_j]}}(h_j(b))$, where $[k_{0|1} \triangleq (i|(1-i))$ and$]$ $h_{0|1} \triangleq (\Delta_{A_{[3,i]}}|(\mu[{\restriction}A_{3,i}])) \in \hom(\mathfrak{A}_{[3,i]},\mathfrak{A})$. In this way, Theorem 4.7 yields (i). Now, assume the ()-non-optional version of (iii) holds. Then, there is some non-diagonal homomorphism h from [a subalgebra of] $\mathfrak{A}$ to $\mathfrak{A}$ with $B \triangleq (\operatorname{img} h)$ not being a singleton, in which case B forms a non-one-element subalgebra of $\mathfrak{A}$, and so does $D \triangleq (\operatorname{dom} h)$. Hence, $\Delta_2 \subseteq (B \cap D)$. Then, both of $(\mathfrak{B}|\mathfrak{D}) \triangleq (\mathfrak{A}{\restriction}(B|D))$ are $(\wedge,\vee)$-lattices with zero/unit $\langle 0/1, 0/1\rangle$, for $\mathfrak{A}$ is so, in which case, as $h \in \hom(\mathfrak{D},\mathfrak{B})$ is surjective, by Lemma 2.1, $h{\restriction}\Delta_2$ is diagonal, and so, since h is not so, there is some $i \in 2$ such that $DM_{3,-,i} \subseteq D$ {in particular, $A_{[3,i]} \subseteq D$}, while $h(\langle 1-i, i\rangle) \neq \langle 1-i, i\rangle$. On the other hand, for all $a \in A$, it holds that $(\sim^{\mathfrak{A}} a = a) \Leftrightarrow (a \notin \Delta_2)$, in which case $\sim^{\mathfrak{A}} h(\langle 1-i,i\rangle) = h(\sim^{\mathfrak{A}}\langle 1-i,i\rangle) = h(\langle 1-i,i\rangle)$, and so $h(\langle 1-i,i\rangle) = \langle i, 1-i\rangle$. And what is more, [if $A_{3,i} = A$, then] $\langle i, 1-i\rangle \in D$, in which case we have $(\langle i,1-i\rangle(\wedge|\vee)^{\mathfrak{D}}\langle 1-i,i\rangle) = \langle 0|1, 0|1\rangle$, and so, by the diagonality of $h{\restriction}\Delta_2$, we get $(h(\langle i,1-i\rangle)(\wedge|\vee)^{\mathfrak{A}}\langle i,1-i\rangle) = (h(\langle i,1-i\rangle)(\wedge|\vee)^{\mathfrak{A}}h(\langle 1-i,i\rangle)) = h(\langle 0|1,0|1\rangle) = \langle 0|1, 0|1\rangle$ (in particular, $h(\langle i,1-i\rangle) = \langle 1-i,i\rangle$). In this way, $\hom(\mathfrak{D},\mathfrak{A}) \ni h = (\mu{\restriction}D)$, in which case, as $A_{[3,i]} \subseteq D$, $(\mu[{\restriction}A_{3,i}]) \in \hom(\mathfrak{A}_{[3,i]},\mathfrak{A})$, and so (ii) holds. Finally, if $\Delta_2\langle = (\bigcap_{i\in 2} DM_{3,-,i}) \subseteq (\bigcap_{i\in 2} A_{3,i})\rangle$ does not form a subalgebra of $\mathfrak{A}$, then there are some $\varsigma \in \Sigma$ of arity $n \in \omega$ and some $\bar{a} \in \Delta_2^n$ such that $b \triangleq \varsigma^{\mathfrak{A}}(\bar{a}) \in (A_{[3,i]} \setminus \Delta_2)$ [where $i \triangleq \pi_1(b) \in 2$], in which case $\mu(b) \neq b = \varsigma^{\mathfrak{A}}(\mu \circ \bar{a})$, and so

$(\mu[\upharpoonright A_{3,i}]) \notin \hom(\mathfrak{A}_{[3,i]}, \mathfrak{A})$. Thus, (ii)⇒(v) is by (2.14), so, as the []-optional version of (ii) is a particular case of the non-[]-optional one, (i)⇔(ii) ends the proof. □

As μ is not diagonal, according to Example 11 of [19], the optional and non-optional versions of the item (ii) of Theorem 6.11 are non-equivalent to one another, and so are those of (i/iii) (in particular, the converse of the final assertion of Theorem 6.11 does not hold). Theorem 6.11(ii)⇒(i) positively covers both $B_{4\{01\}[3]}$ and the purely classically-negative expansion of $B_{4\{01\}}$, the underlying algebra $\mathfrak{DMB}_{4\{01\}}$ of the characteristic matrix of which though has no three-element subalgebra. In view of Theorem 6.11(i)⇒(iv), the self-extensionality of these three instances of uniform four-valued expansions of B_4 provides a new insight and a new proof (convergent with those given by [19]) to the non-algebraizability of the sequent calculi associated (according to [18]) with their characteristic matrices, proved originally in [17] by a quite different (though equally generic) method based upon universal tools elaborated in [16]. This well justifies the thesis of the first paragraph of Section 1. Conversely, using Theorem 6.11(i)⇒(iv) /"and Remark 4.14", we immediately conclude that arbitrary bilattice/implicative uniform four-valued expansions of B_4 /"as well as their double three-valued extensions in the purely implicative case" are not self-extensional, for their /⊃-implicative characteristic matrices have equational "implication $\{(((x_0 \sqcup \sim x_0) \sqcup (x_1 \sqcup \sim x_1)) \wedge x_0) \lessapprox (((x_0 \sqcup \sim x_0) \sqcup (x_1 \sqcup \sim x_1)) \vee x_1)\}$, in view of the proof of Theorem 4.30 of [17]"/"truth definition $\{x_0 \approx (x_0 \supset x_0)\}$". According to Corollary 5.2/5.3 of [23], this does equally/not ensue from Theorem 6.11(i)⇒(v)/", so refuting the inverse".

Finally, since inferentially inconsistent logics are self-extensional, by (2.12), Theorems 6.9, 6.10, 6.11(i)⇔(iii)⇒(iv)[⇒(iii)], Remark 2.5 and Example 4.2, we get:

Theorem 6.12. *Let* M *be a class of no-more-than-four-valued models of* C *and* C' *the logic of* M. *Then,* C' *is self-extensional iff either* M *contains no non-∼-classically-defining truth-non-empty consistent element or there are a non-diagonal non-singular homomorphism from [a subalgebra of]* $\mathfrak{A}$ *to* $\mathfrak{A}$ *[i.e.,* $\mathcal{A}$ *has no equational implication] as well as both ∼-paraconsistent and [truth-non-empty]* $(\vee, \sim)$*-paracomplete [distinct] element[s] of* M. *In particular, any inferentially consistent non-∼-classical no-more-than-four-valued extension of* C *is self-extensional only if it is both ∼-paraconsistent and* $(\vee, \sim)$*-paracomplete.*

6.1.3.1.1. Theorems versus bounds.

Corollary 6.13. *Suppose* C *is self-extensional (i.e.,* μ *is an endomorphism of* $\mathfrak{A}$*; cf. Theorem 6.11(i)⇔(ii)). Then, the following are equivalent:*

(i) *C has a theorem (in particular, is implicative; cf.* (2.5)*);*
(ii) $\top^{\mathfrak{DM}_{4,01}}$ *is term-wise definable in* $\mathfrak{A}$*;*
(iii) $\bot^{\mathfrak{DM}_{4,01}}$ *is term-wise definable in* $\mathfrak{A}$.

Proof. First, assume (i) holds. Then, by Remark 2.4, there is some $\phi \in (C(\varnothing) \cap \mathrm{Fm}^1_\Sigma)$, in which case, by the structurality of C, for each $i \in 2$, $\psi_i \triangleq \phi(x_i) \in C(\varnothing)$, and so, by Remark 4.8 and Theorem 6.11(i)⇒(v), for all $a \in A$, we have $\psi_0^{\mathfrak{A}}(a) = \psi_0^{\mathfrak{A}}[x_0/a, x_1/1] = \psi_1^{\mathfrak{A}}[x_0/a, x_1/1] = \psi_1^{\mathfrak{A}}[x_1/1] \in (\Delta_2 \cap D^{\mathcal{A}}) = \{\mathsf{t}\}$. Thus, (ii) holds. Next, (ii)⇔(iii) is by the fact that $\sim^{\mathfrak{A}}(kk) = ((1-k)(1-k))$, for all $k \in 2$. Finally, (ii)⇒(i) is by the fact that $\mathsf{t} \in D^{\mathcal{A}}$. □

6.1.3.1.2. Implicativity versus maximal paraconsistency.

Theorem 6.14. *Suppose* C *is self-extensional (i.e.,* μ *is an endomorphism of* $\mathfrak{A}$*; cf. Theorem 6.11(i)⇔(ii)). Then, the following are equivalent:*

(i) *$\mathcal{A}$ is implicative (viz., C is so; cf. Lemma 6.1);*
(ii) *$\mathcal{A}$ is negative;*

(iii) $\neg^{\mathfrak{DMB}_4}$ *is term-wise definable in* $\mathfrak{A}$*;*
(iv) $DM_{3,0}$ *does not form a subalgebra of* $\mathfrak{A}$*, and* C *has a theorem;*
(v) $DM_{3,1}$ *does not form a subalgebra of* $\mathfrak{A}$*, and* C *has a theorem;*
(vi) C *is maximally* $\sim$*-paraconsistent and has a theorem;*

In particular, C *is maximally* $\sim$*-paraconsistent, whenever it is both implicative and self-extensional.*

Proof. First, (ii)⇒(i) is by Remark 2.8(i)(b) and the $\vee$-disjunctivity of $\mathcal{A}$. Conversely, if $\mathcal{A}$ is $\sqsupset$-implicative, then, by Corollary 6.13(i)⇒(iii), there is some $\varphi \in \mathrm{Fm}^1_\Sigma$ such that $\varphi^{\mathfrak{A}}(a) = (00)$, for all $a \in A$, in which case $\mathcal{A}$ is $\wr$-negative, where $(\wr x_0) \triangleq (x_0 \sqsupset \varphi)$, and so (ii) holds.

Next, (ii) is a particular case of (iii). Conversely, assume $\mathcal{A}$ is $\wr$-negative. Then, by Theorem 6.11(i)⇒(v), $\wr^{\mathfrak{A}}(ii) = ((1-i)(1-i))$, for each $i \in 2$. And what is more, if, for any $j \in 2$, $\wr^{\mathfrak{A}}(j(1-j))$ was not equal to $((1-j)j)$, then it would be equal to $((1-j)(1-j))$, in which case we would have $((1-j)(1-j)) = \mu((1-j)(1-j)) = \mu(\wr^{\mathfrak{A}}(j(1-j))) = \wr^{\mathfrak{A}}\mu(j(1-j)) = \wr^{\mathfrak{A}}((1-j)j)$, and so would get $(1-j) = \pi_0(\wr^{\mathfrak{A}}((1-j)j) = (1-(1-j)) = j$. In this way, (iii) holds.

Further, (iii)⇒(v) is by (iii)⇒(i), (2.5) and the fact that $\neg^{\mathfrak{DMB}_4}\mathsf{n} = \mathsf{b} \notin DM_{3,1} \ni \mathsf{n}$. Conversely, assume (v) holds. Then, there is some $\phi \in \mathrm{Fm}^3_\Sigma$ such that $\phi^{\mathfrak{A}}(\mathsf{n},\mathsf{t},\mathsf{f}) = \mathsf{b}$. Moreover, by Corollary 6.13(i)⇒(ii), there is some $\psi \in \mathrm{Fm}^1_\Sigma$ such that $\psi^{\mathfrak{A}}(a) = \mathsf{t}$, for all $a \in A$. Let $\xi \triangleq (\phi[x_{i+1}/{\sim}^i\psi]_{i\in 2}) \in \mathrm{Fm}^1_\Sigma$, in which case $\xi^{\mathfrak{A}}(\mathsf{n}) = \mathsf{b}$, and so $\mathsf{n} = \mu(\mathsf{b}) = \xi^{\mathfrak{A}}(\mu(\mathsf{n})) = \xi^{\mathfrak{A}}(\mathsf{b})$. And what is more, by Theorem 6.11(i)⇒(v), $\xi^{\mathfrak{A}}[\Delta_2] \subseteq \Delta_2$. Let $k \triangleq \pi_0(\psi^{\mathfrak{A}}(\mathsf{f})) \in 2$ and $\varphi \triangleq (({\sim}^k\xi \vee {\sim}x_0) \wedge {\sim}^{1-k}\xi) \in \mathrm{Fm}^1_\Sigma$, in which case $\varphi^{\mathfrak{A}} = \neg^{\mathfrak{DMB}_4}$, and so (iii) holds.

Furthermore, (iv)⇔(v) is by the fact that $\mu[DM_{3,l}] = DM_{3,1-l}$, for all $l \in 2$. Finally, (iv)⇔(vi) is due to Theorem 4.31(vi)⇔(i) of [23]. □

6.2. **Uniform three-valued logics with subclassical negation.**

6.2.1. *U3VLSN versus super-classical matrices.* Σ-matrices with $\sim$-reduct having a (canonical) $\sim$-classical submatrix {and so being both consistent and truth-non-empty, for latter ones are so; cf. Remark 2.8(ii)(b)} (and carrier $3 \div 2$; cf. Subparagraph 2.2.1.2.1) are said to be *([3-]canonical⟨ly⟩)* $\sim$*-super-classical*, in which case, by (2.14), $\sim$ is a subclassical negation for their logics {cf. Paragraph 2.3.2.1}, and so we have the "if" part of the following marking the framework of this subsection:

Theorem 6.15. *Let* $\mathcal{A}$ *be a (no-more-than-*$(2[+1])$*-valued)* Σ*-matrix. Then,* $\sim$ *is a subclassical negation for the logic of* $\mathcal{A}$ *if(f)* $\mathcal{A}$ *is* $\sim$*-[super-]classical. In particular, any uniform three-valued* Σ*-logic with subclassical negation* $\sim$ *is minimally so iff it is not* $\sim$*-classical.*

Proof. (Assume $\sim$ is a subclassical negation for the logic of $\mathcal{A}$. First, by (2.16) with $m = 1$ and $n = 0$, there is some $a \in D^{\mathcal{A}}$ such that ${\sim}^{\mathfrak{A}}a \notin D^{\mathcal{A}}$. Likewise, by (2.16) with $m = 0$ and $n = 1$, there is some $b \in (A \setminus D^{\mathcal{A}})$ such that ${\sim}^{\mathfrak{A}}b \in D^{\mathcal{A}}$, in which case $a \neq b$, and so $|A| \neq 1$. Then, if $|A| = 2$, we have $A = \{a, b\}$, in which case $\mathcal{A}$ is $\sim$-classical, and so $\sim$-super-classical. [Now, assume $|A| = 3$.

Claim 6.16. *Let* $\mathcal{A}$ *be a three-valued* Σ*-matrix,* $\bar{a} \in A^2$ *and* $i \in 2$*. Suppose* $\sim$ *is a subclassical negation for the logic of* $\mathcal{A}$*, and, for each* $j \in 2$*,* $(a_j \in D^{\mathcal{A}}) \Leftrightarrow ({\sim}^{\mathfrak{A}}a_j \notin D^{\mathcal{A}}) \Leftrightarrow (a_{1-j} \notin D^{\mathcal{A}})$*. Then, either* ${\sim}^{\mathfrak{A}}a_i = a_{1-i}$ *or* ${\sim}^{\mathfrak{A}}{\sim}^{\mathfrak{A}}a_i = a_i$*.*

Proof. By contradiction. For suppose both ${\sim}^{\mathfrak{A}}a_i \neq a_{1-i}$ and ${\sim}^{\mathfrak{A}}{\sim}^{\mathfrak{A}}a_i \neq a_i$. Then, in case $a_i \in / \notin D^{\mathcal{A}}$, as $|A| = 3$, we have both $(D^{\mathcal{A}}/(A \setminus D^{\mathcal{A}})) = \{a_i\}$, in which case ${\sim}^{\mathfrak{A}}a_{1-i} = a_i$, and $((A \setminus D^{\mathcal{A}})/D^{\mathcal{A}}) = \{a_{1-i}, {\sim}^{\mathfrak{A}}a_i\}$, respectively. Consider the following exhaustive cases:

- $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}}a_i = a_{1-i}$. Then, $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}}\sim^{\mathfrak{A}}a_i = a_i$. This contradicts to (2.16) with $(n/m) = 0$ and $(m/n) = 3$, respectively.
- $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}}a_i = \sim^{\mathfrak{A}}a_i$. Then, for each $c \in ((A \setminus D^{\mathcal{A}})/D^{\mathcal{A}})$, $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}}\sim^{\mathfrak{A}}c = \sim^{\mathfrak{A}}a_i \notin / \in D^{\mathcal{A}}$. This contradicts to (2.16) with $(n/m) = 3$ and $(m/n) = 0$.

Thus, in any case, we come to a contradiction, as required. □

Set $d_0 \triangleq a$ and $d_1 \triangleq b$. Consider the following complementary cases:

- for each $k \in 2$, $\sim^{\mathfrak{A}}d_k = d_{1-k}$. Then, $\{a, b\}$ forms a subalgebra of $\mathfrak{A}\restriction\{\sim\}$, $(\mathcal{A}\restriction\{\sim\})\restriction\{a, b\}$ being a $\sim$-classical submatrix of $\mathcal{A}\restriction\{\sim\}$, as required.
- for some $k \in 2$, $\sim^{\mathfrak{A}}d_k \neq d_{1-k}$, in which case, by Claim 6.16, $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}}d_k = d_k$, so $\{d_k, \sim^{\mathfrak{A}}d_k\}$ forms a subalgebra of $\mathfrak{A}\restriction\{\sim\}$, $(\mathcal{A}\restriction\{\sim\})\restriction\{d_k, \sim^{\mathfrak{A}}d_k\}$ being a $\sim$-classical submatrix of $\mathcal{A}\restriction\{\sim\}$, as required.]) □

The "only if" part of Theorem 6.15 does not, generally speaking, hold for no-less-than-four-valued logics, in view of:

Example 6.17. Let $n \in \omega$ and $\mathcal{A}$ any Σ-matrix with $A \triangleq (n \cup (2 \times 2))$, $D^{\mathcal{A}} \triangleq \{\langle 1, 0\rangle, \langle 1, 1\rangle\}$, $\sim^{\mathfrak{A}}\langle i, j\rangle \triangleq \langle 1 - i, (1 - i + j) \mod 2\rangle$, for all $i, j \in 2$, and $\sim^{\mathfrak{A}}k \triangleq \langle 1, 0\rangle$, for all $k \in n$. Then, for any subalgebra $\mathfrak{B}$ of $\mathfrak{A}\restriction\{\sim\}$, we have $(2 \times 2) \subseteq B$, in which case $4 \leqslant |B|$, and so $\mathcal{A}$ is not $\sim$-super-classical, for $4 \nleqslant 2$. On the other hand, 2×2 forms a subalgebra of $\mathfrak{A}\restriction\{\sim\}$, while $\mathcal{B} \triangleq (\mathcal{A}\restriction\{\sim\})\restriction(2\times 2)$ is $\sim$-negative, in which case $\theta^{\mathcal{B}} \in \mathrm{Con}(\mathfrak{B})$, and so $h \triangleq \chi^{\mathcal{B}}$ is a surjective strict homomorphism from $\mathcal{B}$ onto the classically-canonical (in particular, two-valued) $\{\sim\}$-matrix $\mathcal{C} \triangleq \langle h[\mathfrak{B}], \{1\}\rangle$, (in particular, by Remark 2.8(ii)(a), $\mathcal{C}$ is $\sim$-classical, so, by (2.14), $\sim$ is a subclassical negation for the logic of $\mathcal{A}$). □

Likewise, U3VLSN need not be minimally so, in view of Example 6.21 below.

In general, given any three-valued $\sim$-super-classical Σ-matrix $\mathcal{A}$ with $\sim$-classical submatrix $\mathcal{B}$ of its $\sim$-reduct, the bijective mapping $e \triangleq (\chi^{\mathcal{B}} \cup ((A \setminus B) \times \{\frac{1}{2}\}) : A \to (3 \div 2)$ is an isomorphism from $\mathcal{A}$ onto the canonical $\sim$-super-classical Σ-matrix $\complement_{[3]}(\mathcal{A}) \triangleq \langle e[\mathfrak{A}], e[D^{\mathcal{A}}]\rangle$, called the *[3-]canonization of* $\mathcal{A}$.

Throughout the rest of this subsection, unless otherwise specified, C is supposed to be the logic of an arbitrary but fixed canonical $\sim$-super-classical Σ-matrix $\mathcal{A}$ (that exhausts all uniform three-valued Σ-logics with subclassical negation $\sim$, in view of Theorem 6.15 and (2.14)), in which case this is false-singular iff it is not truth-singular iff $\Bbbk^{\mathcal{A}} \triangleq \chi^{\mathcal{A}}(\frac{1}{2}) = 1$, and so is false-/truth-singular, whenever it is $\sim$-paraconsistent/"both weakly $\veebar$-disjunctive and $(\veebar, \sim)$-paracomplete", respectively, in which case C is not $\sim$-classical, in view of Remark 2.8(i)(c)/(d). And what is more, any proper submatrix $\mathcal{B}$ of $\mathcal{A}$ is either $\sim$-classical or one-valued, in which case $\mathcal{B}$ is simple, and so $\mathcal{A}$ is simple iff it is hereditarily so. Also, $\mathcal{A}$ is [weakly] $\diamond$-conjunctive/-disjunctive/-implicative iff C is so, in view of the following results:

Lemma 6.18. *Let $\mathcal{B}$ be a Σ-matrix and C' the logic of $\mathcal{B}$. Suppose $\mathcal{B}$ is [not] false-singular [as well as both no-more-than-three-valued and $\sim$-super-classical]. Then, the following are equivalent:*

(i) *C' is $\veebar$-disjunctive;*
(ii) *$\mathcal{B}$ is $\veebar$-disjunctive;*
(iii) (2.2) *with $i = 0$,* (2.3) *and* (2.4) *[as well as* (3.2)*] are satisfied in C'.*

Proof. First, (ii)⇒(i) is immediate. Next, assume (i) holds. Then, (2.2) with $i = 0$, (2.3) and (2.4) are immediate. [And what is more, once $\mathcal{B}$ is not false-singular, it is both no-more-than-three-valued (and so truth-singular) and $\sim$-super-classical, in which case it is not $\sim$-paraconsistent, and so is C'. Then, by (i) and Lemma 3.11, (3.2) is satisfied in C'.] Thus, (iii) holds. Finally, assume (iii) holds. Consider any $a, b \in B$. Then, by (2.2) with $i = 0$ and (2.3), C' is weakly $\veebar$-disjunctive, and so is $\mathcal{B}$, in which case $(a \veebar^{\mathfrak{B}} b) \in D^{\mathcal{B}}$, whenever either a or b is in $D^{\mathcal{B}}$. Now, assume $(\{a, b\} \cap D^{\mathcal{B}}) = \varnothing$. Then, in case $a = b$ (in particular, $\mathcal{B}$ is false-singular), by (2.4), we get $D^{\mathcal{B}} \not\ni (a \veebar^{\mathfrak{B}} a) = (a \veebar^{\mathfrak{B}} b)$. [Otherwise, $\mathcal{B}$ is not false-singular, in which case it is no-more-than-three-valued (in particular, truth-singular) and $\sim$-super-classical, while (3.2) is true in $\mathcal{B}$, and so, for some $c \in (B \setminus D^{\mathcal{B}}) = \{a, b\}$, it holds that $\sim^{\mathfrak{B}} c \in D^{\mathcal{B}}$, while $\sim^{\mathfrak{B}}\sim^{\mathfrak{B}} c = c$. Let d be the unique element of $\{a, b\} \setminus \{c\}$, in which case $\{a, b\} = \{c, d\}$. Then, since $\sim^{\mathfrak{B}} c \in D^{\mathcal{B}}$, we conclude that $(c \veebar^{\mathfrak{B}} d) = (\sim^{\mathfrak{B}}\sim^{\mathfrak{B}} c \veebar^{\mathfrak{B}} d) \notin D^{\mathcal{B}}$, for, otherwise, by (2.2) with $i = 0$ and (3.2), we would get $d \in D^{\mathcal{B}}$. Hence, by (2.3), we eventually get $(a \veebar^{\mathfrak{B}} b) \notin D^{\mathcal{B}}$.] □

Corollary 6.19. *[Providing $\mathcal{A}$ is false-singular (in particular, $\sim$-paraconsistent)] $\mathcal{A}$ is $\sqsupset$-implicative iff C is [weakly] so.*

Proof. By (2.6), (2.7), (2.9), Theorem 3.5(ii) and Lemma[s 3.14 and] 6.18. □

Remark 6.20. $\mathcal{A}$ is not $\sim$-negative iff it has unitary equality determinant $\{x_0, \sim x_0\}$.

Next, $\mathcal{A}$ is said to be *($\sim$-)involutive*, provided $\sim^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2}$, that is, the Σ-identity $\sim\sim x_0 \approx x_0$ is true in $\mathfrak{A}$, in which case $\mathcal{A}$ is not $\sim$-negative. Further, $\mathcal{A}$ is said to be *[extra-]classically-hereditary*, provided $[A\backslash]2$ forms a subalgebra of $\mathfrak{A}$ [in which case $\mathcal{A}$ is involutive]. Then, $\mathcal{A}$, being classically-hereditary, is said to be *genuinely*|*"[weakly] $\diamond$-conjunctively/-disjunctively/-implicatively"* so, whenever $\mathcal{A}{\restriction}2$ is "genuinely $\sim$-classical"|"[weakly] $\diamond$-conjunctive/-disjunctive/-implicative", respectively. Likewise, $\mathcal{A}$ is said to be *quadro-classically hereditary*, whenever $L_4 \triangleq (A^2 \setminus (2^2 \cup \Delta_A))$ forms a subalgebra of $\mathfrak{A}^2$, in which case $\mathcal{A}$ is involutive, and so $\mathcal{A}^2{\restriction}L_4$ is $\sim$-negative, whenever $\mathcal{A}$ is false-singular. Finally, $\mathcal{A}$ is said to be *classically-valued*, provided, for all $\varsigma \in \Sigma$, $(\operatorname{img} \varsigma^{\mathfrak{A}}) \subseteq 2$, in which case $\mathcal{A}$ is [not extra-]classically-hereditary [more specifically, not involutive].

6.2.1.1. Examples.

6.2.1.1.1. Kleene-style logics. Let $\Sigma \triangleq \Sigma_{\sim,+[01]}$ and $\mathcal{A}$ both involutive and truth-/false-singular with $(\mathfrak{A}{\restriction}\Sigma_{+[01]}) \triangleq \mathfrak{D}_{3[01]}$. Then, $\mathcal{A}$ is both $\wedge$-conjunctive, $\vee$-disjunctive and non-$\sim$-negative, in which case it is $(\vee, \sim)$-paracomplete/$\sim$-paraconsistent, and so, by Remark 2.8(i)(c)/(d), C is not $\sim$-classical, as well as both classically-hereditary and [not] extra-classically-hereditary, while $\mathfrak{A}$ is a distributive $(\wedge, \vee)$-lattice with zero 0 and unit 1, whereas C is [the *bounded version*|*expansion* $KL_{3,01}$/ LP_{01} *of*] "Kleene's three-valued logic"/"the *logic of paradox*" KL_3/LP [6]/[13].

6.2.1.1.2. Gödel-style logics. Let $\Sigma \triangleq \Sigma^{/\supset}_{\sim,+,01}$ and $\mathcal{A}$ [not] truth-singular as well as neither $\sim$-negative nor involutive with $(\mathfrak{A}{\restriction}\Sigma_{+,01}) \triangleq \mathfrak{D}_{3,01}$ (in which case $\sim^{\mathfrak{A}}$ is the [dual] pseudo-complement operation)/" as well as $\supset^{\mathfrak{A}}$ being the [dual] relative pseudo-complement operation". Then, $\mathcal{A}$ is both $\wedge$-conjunctive, $\vee$-disjunctive and [not] $(\vee, \sim)$-paracomplete as well as [not] non-$\sim$-paraconsistent, and so, by Remark 2.8(i)(c,(d)), C is not $\sim$-classical, while $\mathcal{A}$ is classically-hereditary but not extra-classically-hereditary, whereas C is [the *($\sim$-)paraconsistent counterpart* $PG_3^{*/}$ *of*] "the implication-less fragment G_3^* of"/ Gödel's three-valued logic G_3 [3].

6.2.1.1.3. Hałkowska-Zajac' logic. Let $\Sigma \triangleq \Sigma_{\sim,+}$ and $\mathcal{A}$ both false-singular and involutive with $\mathfrak{A}$ being the distributive $(\wedge, \vee)$-lattice with zero $\frac{1}{2}$ and unit 1. Then, $\mathcal{A}$ is $\sim$-paraconsistent (in particular, C is not $\sim$-classical; cf. Remark 2.8(i)(c)) as well as both classically- and extra-classically-hereditary but weakly neither $\wedge$-conjunctive nor $\vee$-disjunctive, C being the logic HZ [5]. On the other hand, since the identity $\sim\sim x_0 \approx x_0$ is true in $\mathfrak{A}$, $\mathfrak{A}$ is a distributive $(\vee^{\sim}, \wedge^{\sim})$-lattice (cf. Remark 2.8(i)(a) for definition of these secondary binary connectives) with zero $\sim^{\mathfrak{A}}1 = 0$ and unit $\sim^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2}$. Then, $\mathcal{A}$ is both $\vee^{\sim}$-conjunctive and $\wedge^{\sim}$-disjunctive.

6.2.1.1.4. Sette-style logics. Let $\Sigma \triangleq \Sigma^{\supset}_{\sim}$ and $\mathcal{A}$ classically-valued, non-$\sim$-negative, $\supset$-implicative (in particular, $\uplus_{\supset}$-disjunctive) and [not] false-singular. Then, $\mathcal{A}$ is

[not] $\sim$-paraconsistent as well as [not] non-$(\uplus_{\supset}, \sim)$-paracomplete, and so, by Remark 2.8(i)(c,d), C, being [the *intuitionistic/($(\uplus_{\supset}, \sim)$-)paracomplete counterpart* IP^1 *of*] P^1 [28], is not $\sim$-classical.

6.2.2. *Minimal U3VLSN.* Let $\Delta_2^+ \triangleq \Delta_2 \in 2^2$ and $\Delta_2^- \triangleq (A^2 \setminus \Delta_2) \in 2^2$.

Generally speaking, C, though being three-valued, need not be minimally uniformly three-valued (viz., non-$\sim$-classical), in view of:

Example 6.21. Let $\Sigma \triangleq \Sigma_{\sim[,+,01]}$ and $(\mathcal{B}/\mathcal{D})|\mathcal{E}$ the [$\wedge$-conjunctive $\vee$-disjunctive] canonical "$\sim$-negative false-/truth-singular $\sim$-super-classical"|$\sim$-classical Σ-matrix [with $(((\mathfrak{B}/\mathfrak{D})|\mathfrak{E})\restriction\Sigma_+) \triangleq \mathfrak{D}_{3|2}$ (cf. Subparagraph 2.2.1.2.1), $\bot^{((\mathfrak{B}/\mathfrak{D})|\mathfrak{E}} \triangleq ((0/\frac{1}{2})|0)$ and $\top^{((\mathfrak{B}/\mathfrak{D})|\mathfrak{E}} \triangleq ((\frac{1}{2}/1)|1)$, respectively, in which case $(\mathcal{B}/\mathcal{D})|\mathcal{E}$ has tautology $\top$ and, in view of Remark 2.8(i)(b), is $\sqsupset^{\sim}_{\vee}$-implicative]. On the other hand, in the non-optional case, Δ_2^- forms a subalgebra of $(\mathfrak{B}/\mathfrak{D})^2$, in which case, by (2.14), $(\mathcal{B}/\mathcal{D})^2\restriction\Delta_2^-$ is a truth-empty model of the logic of $\mathcal{B}/\mathcal{D}$, and so, by Corollary 3.10(ii)$\Rightarrow$(i), this has no tautology. Then, $\chi^{\mathcal{B}/\mathcal{D}} \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{B}/\mathcal{D}, \mathcal{E})$. Therefore, by (2.14), $\mathcal{B}/\mathcal{D}$ define the same $\sim$-classical Σ-logic of $\mathcal{E}$, in which case, by Remark 2.8(i)(c), this is non-$\sim$-paraconsistent, and so is any extension of it. And what is more, by Remark 2.8(ii)(c), $\mathcal{B}$ and $\mathcal{D}$ are non-isomorphic [as well as $\mathfrak{B}$ and $\mathfrak{D}$ are so, because the Σ-identity $(x_0 \wedge \sim x_0) \approx \sim\sim\bot$, being true in $\mathfrak{B}$, is not so in $\mathfrak{D}$ under $[x_0/\frac{1}{2}]$], while $h : (B/D) \to (B/D) : a \mapsto (\max\min)(0/1, \chi^{\mathcal{B}/\mathcal{D}}(a) - / + \frac{1}{2})$ is a non-diagonal (for $h(1/0) = \frac{1}{2} \neq (1/0)$) strict homomorphism from $\mathcal{B}/\mathcal{D}$ to itself, so this does not have a unitary equality determinant, in view of Theorem 3.3, whereas $[(\top/\bot)^{\mathfrak{B}/\mathfrak{D}} = \frac{1}{2} \notin 2$, in which case] $\mathcal{B}/\mathcal{D}$, being $\sim$-negative (and so non-involutive), is not quadro-classically hereditary [as well as not classically so]. $\Box$

On the other hand, $\sim$-classical Σ-logics are self-extensional, in view of Example 4.2. This makes the purely algebraic criterion of the minimality of U3VLSN to be obtained here especially acute.

Lemma 6.22 (Key 3-valued Lemma)**.** *Let $\mathcal{B}$ be a canonical $\sim$-super-classical Σ-matrix, $\mathcal{D}$ a submatrix of $\mathcal{A}$ and $h \in \hom(\mathfrak{D}, \mathfrak{B})$. Then, providing $\mathcal{A}$ is involutive, whenever both $\mathcal{B}$ is so and $\frac{1}{2} \in (\operatorname{img} h)$ (in particular, either $\mathfrak{A} = \mathfrak{B}$ or $\hom(\mathfrak{B}\restriction(\operatorname{img} h), \mathfrak{A}) \neq \varnothing$), the following hold* {*cf. p. 2*}*:*

(i) *providing h is not singular, $2 \subseteq D$, while $h[2] = 2$, in which case $h\restriction 2$ is injective, and so belongs to $\{\Delta_2^+, \Delta_2^-\}$;*
(ii) *providing $h \not\supseteq \Delta_2^-$ [in particular, $h \in \hom(\mathcal{D}, \mathcal{B})$] is injective, it is diagonal.*

In particular, the following hold:

(a) *any partial automorphism* {*cf. Subsubsection 3.1.1*} *of $\mathcal{A}$ is diagonal;*
(b) *any isomorphism from $\mathcal{A}$ onto $\mathcal{B}$ is diagonal, in which case $\mathcal{A} = \mathcal{B}$, and so $\mathcal{A}$ and $\mathcal{B}$ are equal, whenever they are isomorphic.*

Proof. First, note that the carrier of any subalgebra of $(\mathfrak{A}|\mathfrak{B})\restriction\{\sim\}$ (in particular, $D|(\operatorname{img} h)$) belongs to $\{A|B, 2, \{\frac{1}{2}\}\}$. And what is more, for each $a \in (A|B)$, we have $(\sim^{\mathfrak{A}|\mathfrak{B}} a = a) \Rightarrow (a = \frac{1}{2})$. In particular, for any $g \in \hom(\mathfrak{D}|(\mathfrak{B}\restriction(\operatorname{img} h)), \mathfrak{B}|\mathfrak{A})$ with $\frac{1}{2} \in (\operatorname{dom} g)$, providing $\sim^{\mathfrak{A}|\mathfrak{B}} \frac{1}{2} = \frac{1}{2}$, we have $\sim^{\mathfrak{B}|\mathfrak{A}} g(\frac{1}{2}) = g(\frac{1}{2})$, in which case we get $g(\frac{1}{2}) = \frac{1}{2}$, and so $\sim^{\mathfrak{B}|\mathfrak{A}} \frac{1}{2} = \frac{1}{2}$. While proving (i,ii), assume $(\sim^{\mathfrak{B}} \frac{1}{2} = \frac{1}{2}) \Rightarrow (\sim^{\mathfrak{A}} \frac{1}{2} = \frac{1}{2})$, whenever $\frac{1}{2} \in (\operatorname{img} h)$.

(i) Assume h is not singular, in which case $1 < |\operatorname{img} h| \leqslant |D|$, and so $D \supseteq 2 \subseteq (\operatorname{img} h)$. Then, as 2 forms a subalgebra of $\mathfrak{A}\restriction\{\sim\}$, $h[2]$ forms a no-more-than-two-element subalgebra of $\mathfrak{B}\restriction\{\sim\}$, in which case $h[2] \in \{2, \{\frac{1}{2}\}\}$, and so $h[2] = 2$, for, otherwise, we would have both $(\operatorname{img} h) = h[D] \supseteq h[2] = \{\frac{1}{2}\} \ni \frac{1}{2}$ and $\sim^{\mathfrak{B}} \frac{1}{2} = \frac{1}{2}$, in which case we would get $\sim^{\mathfrak{A}} \frac{1}{2} = \frac{1}{2}$ as well as, since

$|\operatorname{img} h| \neq 1$, both $\frac{1}{2} \in D = (\operatorname{dom} h)$ and $h(\frac{1}{2}) \in 2$, and so would eventually get $2 \ni h(\frac{1}{2}) = \frac{1}{2}$.

(ii) Assume h is injective, while $\{h \in \hom(\mathcal{D}, \mathcal{B})$, in which case $\Delta_2^- \ni \langle 1, 0\rangle \notin h$, for $(1|0) \in | \notin D^{\mathcal{A}|\mathcal{B}}$, and so$\}$ $\Delta_2^- \nsubseteq h$. Then, $h : D \to (\operatorname{img} h)$ is bijective. Therefore, in case h is singular, we have $(\operatorname{img} h) = \{\frac{1}{2}\} = D$, and so $h = \{\langle \frac{1}{2}, \frac{1}{2}\rangle\}$ is diagonal. Otherwise, by (i), $2 \subseteq D$, while $(h{\restriction}2) \subseteq h$ is diagonal. In particular, $h = (h{\restriction}2)$ is diagonal, whenever $D = 2$. Otherwise, $D = A$, while $\frac{1}{2} \notin 2$, in which case, by the injectivity of h, we have $h(\frac{1}{2}) \notin h[2] = 2$, and so we get $h(\frac{1}{2}) = \frac{1}{2}$ (in particular, h is diagonal).

Then, (a/b) is by (ii) with $(\mathcal{B}/\mathcal{D}) = \mathcal{A}$ and and /bijective $h \in \hom(\mathcal{D}, \mathcal{B})$ /"as well as $h^{-1} \in \hom(\mathfrak{B}, \mathfrak{A})$". □

Corollary 6.23. *The following are equivalent:*

(i) *$\mathcal{A}$ has no [unitary] equality determinant;*

(ii) *$\mathcal{A}$ is a strictly (surjectively) homomorphic counter-image of a $\sim$-classical Σ-matrix;*

(iii) *$\mathcal{A}$ is not $\{$hereditarily$\}$ simple;*

(iv) *$\theta^{\mathcal{A}} \in \operatorname{Con}(\mathfrak{A})$ $\langle$in which case $\chi^{\mathcal{A}}$ is a strict surjective homomorphism from $\mathcal{A}$ onto $\mathcal{C}_{\mathcal{A}} \triangleq \langle \chi^{\mathcal{A}}[\mathcal{A}], \{1\}\rangle$, being, in its turn, canonically $\sim$-classical$\rangle$.*

Proof. First, (i)⇔(iii) is by Lemmas 3.1, 6.22(a) and Theorem 3.3.

Next, (ii)⇒(iii) is by Remark 2.6(i,ii), for $|A| = 3 \nleq 2$.

Further, (iii)⇒"$\theta^{\mathcal{A}} \in \operatorname{Con}(\mathfrak{A})$" is by the fact $\operatorname{img}[\theta^{\mathcal{A}} \setminus \Delta_A] = \{\{\frac{1}{2}, \Bbbk^{\mathcal{A}}\}\}$ is a singleton.

Finally, assume $\theta^{\mathcal{A}} \in \operatorname{Con}(\mathfrak{A})$, in which case $h \triangleq \chi^{\mathcal{A}}$ is a strict surjective homomorphism from $\mathcal{A}$ onto the classically-canonical (in particular, two-valued) Σ-matrix $\mathcal{C}_{\mathcal{A}}$, and so $h{\restriction}2$, being diagonal, is a strict surjective homomorphism from the $\sim$-negative Σ-matrix $(\mathcal{A}{\restriction}\{\sim\}){\restriction}2$ onto $\mathcal{C}_{\mathcal{A}}{\restriction}\{\sim\}$. Then, by Remark 2.8(ii)(a), $\mathcal{C}_{\mathcal{A}}{\restriction}\{\sim\}$ is $\sim$-negative, and so is $\mathcal{C}_{\mathcal{A}}$, in which case this is canonically $\sim$-classical. Thus, the optional part of (iv) holds, and so does (ii). □

Next, a $(2([+1]))$-*ary semi-conjunction for/of* a canonical $\sim$-(super-)classical Σ-matrix $\mathcal{B}$ is any $\varphi \in \mathrm{Fm}_{\Sigma}^{2([+1])}$ such that $\varphi^{\mathfrak{B}}(0|1, 1|0([, \frac{1}{2}])) = | \neq (0|1)$. (Clearly, any binary semi-conjunction for $\mathcal{A}$ is a ternary one. Likewise, providing $\mathcal{A}$ is classically hereditary, any binary semi-conjunction for $\mathcal{A}{\restriction}2$ is that for $\mathcal{A}$.) Further, a *ternary [anti-]equalizer for/of* $\mathcal{A}$ is any $\tau \in \mathrm{Fm}_{\Sigma}^3$ such that $\tau^{\mathfrak{A}^2}(\langle 0, 1[-1]\rangle, \langle 1, 0[+1]\rangle, \langle 1, \frac{1}{2}\rangle) \in ([2^2\setminus]\Delta_A)$. (Clearly, any binary semi-conjunction for $\mathcal{A}$ is a ternary equalizer for it. Likewise, $\sim x_2$ is a ternary equalizer for $\mathcal{A}$, whenever $\sim^{\mathfrak{A}}\frac{1}{2} = 0$. Furthermore, if $\mathcal{A}$ is both truth-singular, classically hereditary and $\sqsupset$-implicative, then $x_2 \sqsupset x_0$ is a ternary anti-equalizer for it.) Finally, a *quasi-negation for/of* $\mathcal{A}$ is any $\kappa \in \mathrm{Fm}_{\Sigma}^1$ such that $\kappa^{\mathfrak{A}}[\{\frac{1}{2}, 1\}] \subseteq \{0, \frac{1}{2}\}$. (Clearly, $\sim x_0$ is a quasi-negation for $\mathcal{A}$, whenever this is either involutive or both false-singular and $\sim$-negative.)

Lemma 6.24. *Let $\mathcal{B}$ be a canonically $\sim$-[super-]classical Σ-matrix, I a finite set, $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{B})^I$ and $\mathcal{D}$ a subdirect product of it. Then, the following hold:*

(i) *providing [in case $\mathcal{B}$ is $\sim$-paraconsistent but not weakly conjunctive, both $\mathcal{B}$ is classically hereditary but not extra-classically hereditary, and either $\mathcal{D}$ is $\sim$-negative or either $\mathcal{B}$ has a binary semi-conjunction or both $\mathcal{D}$ is truth-non-empty and $\mathcal{B}$ has either a quasi-negation or ternary equalizer, as well as] $\mathcal{D}$ is truth-non-empty [unless $\mathcal{B}$ is $\sim$-paraconsistent], $(I \times \{j\}) \in D$, for some/each $j \in 2$;*

(ii) *providing $I \neq \varnothing$ (in particular, $\mathcal{D}$ is consistent) as well as, for some $j \in 2$, $(I \times \{j\}) \in D$, for each $\Sigma' \subseteq \Sigma$, $\{\langle a, I \times \{a\}\rangle \mid a = \varphi^{\mathfrak{B}}(0,1), \varphi \in$*

$(\mathrm{Var}_2[\cup \mathrm{Fm}^2_{\Sigma'}])\}$ *is an embedding of [the submatrix of]* $\mathcal{B}{\restriction}\Sigma'$ *[generated by* 2*] into* $\mathcal{D}{\restriction}\Sigma'$.

Proof. Clearly, if $(I \times \{j\}) \in D$, for some $j \in 2$, then, $D \ni \sim^{\mathfrak{D}}(I \times \{j\}) = (I \times \{1-j\})$, in which case, as $2 = \{j, 1-j\}$, $(I \times \{k\}) \in D$, for each $k \in 2$, and so (ii) as well as, since $2 \neq \varnothing$, the equivalence of alternatives in (i) hold. For proving the former alternative in (i), consider the following complementary cases with using Remark 2.8(i)(c) tacitly:

- $\mathcal{B}$ is $\sim$-paraconsistent, in which case it is false-singular, and so $D^{\mathcal{A}} = \{\frac{1}{2}, 1\}$. Consider the following complementary subcases:
  - $\mathcal{B}$ is weakly conjunctive, in which case, by Lemma 3.13, $(I \times \{0\}) \in D$.
  - $\mathcal{B}$ is not weakly conjunctive, in which case it is classically hereditary but not extra-classically hereditary, and so there is some $\psi \in \mathrm{Fm}^1_\Sigma$ such that $\psi^{\mathfrak{B}} : B \to 2$, while either $\mathcal{D}$ is $\sim$-negative or $\mathcal{B}$ has a binary semi-conjunction or both $\mathcal{D}$ is truth-non-empty and $\mathcal{B}$ has either a quasi-negation or a ternary equalizer. Take any $b \in D \neq \varnothing$, in which case $c \triangleq \psi^{\mathfrak{D}}(b) \in (D \cap 2^I)$. Let $J \triangleq \{i \in I \mid \pi_i(c) = 1\}$ and $(l|m|n) \triangleq \psi^{\mathfrak{B}}(0|1|\frac{1}{2}) \in 2$. Consider the following complementary subsubcases:
    - ∗ $\mathcal{B}$ has a binary semi-conjunction ϕ, in which case $D \ni \phi^{\mathfrak{D}}(c, \sim^{\mathfrak{D}} c) = (I \times \{0\})$.
    - ∗ $\mathcal{B}$ has no binary semi-conjunction, in which case $l \neq m$, for, otherwise, $\sim^l \psi$ would be a binary semi-conjunction for $\mathcal{B}$, and so $\{l, m\} = 2 \ni n$. Consider the following complementary subsubsubcases:
      - · either of $J/(I \setminus J)$ is empty, in which case $D \ni c = (I \times \{0/1\})$.
      - · $J \neq \varnothing \neq (I \setminus J)$, in which case, as $0 \notin D^{\mathcal{B}}$, $D \ni c \notin D^{\mathcal{D}} \not\ni \sim^{\mathfrak{D}} c \in D$, and so $\mathcal{D}$ is not $\sim$-negative. Then, $\mathcal{D}$ is truth-non-empty, while $\mathcal{B}$ has either a quasi-negation or a ternary equalizer. Take any $d \in D^{\mathcal{D}} = (D \cap \{\frac{1}{2}, 1\}^I) \neq \varnothing$. Consider the following complementary (for $n \in 2 = \{l, m\}$) subsubsubsubcases:
        - ⋆ $n = m$, in which case $D \ni \psi^{\mathfrak{D}}(d) = (I \times \{n\})$.
        - ⋆ $n = l$. Consider the following complementary subsubsubsubsubcases:
          - ∘ $\mathcal{B}$ has a quasi-negation κ. Then, $D \ni \psi^{\mathfrak{D}}(\kappa^{\mathfrak{D}}(d)) = (I \times \{n\})$.
          - ∘ $\mathcal{B}$ has no quasi-negation, in which case it has a ternary equalizer τ, and so $D \ni e \triangleq \tau^{\mathfrak{D}}(c, \sim^{\mathfrak{D}} c, d) = (I \times \{\flat\})$, for some $\flat \in B$. Let $\ell \triangleq \psi^{\mathfrak{B}}(\flat) \in 2$. Then, $D \ni \psi^{\mathfrak{D}}(e) = (I \times \{\ell\})$.
- $\mathcal{B}$ is not $\sim$-paraconsistent, in which case $\mathcal{D}$ is truth-non-empty, and so there is some $e \in D^{\mathcal{D}} = (D \cap (D^{\mathcal{B}})^I)$. Consider any $i \in I$ and the following complementary subcases:
  - $\mathcal{B}$ is truth-singular, in which case $\pi_i(e) = 1$, and so $\pi_i(\sim^{\mathfrak{D}} e) = \sim^{\mathfrak{B}} \pi_i(e) = 0$.
  - $\mathcal{B}$ is not truth-singular, in which case it is false-singular, and so, as $\pi_i(\sim^{\mathfrak{D}} e) = \sim^{\mathfrak{B}} \pi_i(e) \notin D^{\mathcal{B}}$, for, otherwise, (2.10) would not be true in $\mathcal{B}$ under $[x_0/\pi_i(e), x_1/0]$, we have $\pi_i(\sim^{\mathfrak{D}} e) = 0$.

  Thus, in any case, $\pi_i(\sim^{\mathfrak{D}} e) = 0$, and so $D \ni \sim^{\mathfrak{D}} e = (I \times \{0\})$. □

Let $h_{+/2} : 2^2 \to A, \langle i, j\rangle \mapsto \frac{i+j}{2}$.

Theorem 6.25. *C is $\sim$-classical (viz., non-minimally uniformly three-valued; cf. Theorem 6.15) iff either of the following holds:*

(i) $\theta^{\mathcal{A}} \in \mathrm{Con}(\mathfrak{A})$ *(i.e., $\mathcal{A}$ "has no {unitary} equality determinant"|"is not ⟨hereditarily⟩ simple"|"is a strictly ⌈surjectively⌉ homomorphic counter-image of a $\sim$-classical Σ-matrix) [in which case $\mathcal{C}_{\mathcal{A}} \triangleq \langle \chi^{\mathcal{A}}[\mathfrak{A}], \{1\}\rangle$ is a canonical $\sim$-classical Σ-matrix, being a strictly surjectively homomorphic image of $\mathcal{A}$, and so defines C];*

(ii) *$\mathcal{A}$ is both truth-singular and classically hereditary, while $h_{+/2} \in \mathrm{hom}((\mathfrak{A}{\restriction}2)^2, \mathfrak{A})$ [in which case $h_{+/2} \in \mathrm{hom}^{\mathrm{S}}_{\mathrm{S}}((\mathcal{A}{\restriction}2)^2, \mathcal{A})$, and so $\mathcal{A}{\restriction}2$ is a canonical $\sim$-classical Σ-matrix defining C, whereas $\mathcal{A}$ is neither conjunctive nor disjunctive].*

Proof. Assume both C is $\sim$-classical, in which case, by (2.14), C is defined by a canonical $\sim$-classical (and so both simple and having no proper submatrix) Σ-matrix $\mathcal{B}$, and $\theta^{\mathcal{A}} \notin \mathrm{Con}(\mathfrak{A})$, in which case, by Corollary 6.23(iii)⇒(iv), $\mathcal{A}$ is hereditarily simple, and so, by Lemma 3.7 with $\mathsf{M} = \{\mathcal{B}|\mathcal{A}\}$, there is some finite set $I|J$, some $\overline{\mathcal{C}|\mathcal{D}} \in \mathbf{S}_*(\mathcal{B}|\mathcal{A})^{I|J}$ some subirect product $\mathcal{E}|\mathcal{F}$ of it and some $(h|g) \in \mathrm{hom}^{\mathrm{S}}_{\mathrm{S}}(\mathcal{E}|\mathcal{F}, \mathcal{A}|\mathcal{B})$ (in particular, $\mathfrak{A}|\mathfrak{B}$ belongs to the variety generated by $\mathfrak{B}|\mathfrak{A}$). Then, $\mathcal{A}$ is truth-singular (in particular, non-$\sim$-paraconsistent), for $\mathcal{B}$ is so, while truth-singularity is clearly preserved under $\mathbf{P}$ as well as under both $\mathbf{S}$ and $\mathbf{H}$, in view of Remark 2.8(ii)(c). And what is more, by Remark 2.8(ii)(b), $\mathcal{E}|\mathcal{F}$ is both truth-non-empty and consistent, for $\mathcal{A}|\mathcal{B}$ is so. Then, by Lemma 6.24(i) with $j = (0[+1])$, $(E|F) \ni (a|b)[']\triangleq ((I|J)\times\{j\})$. Let $\mathcal{G}$ be the submatrix of $\mathcal{A}$ generated by 2, in which case it is simple, for $\mathcal{A}$ is hereditarily so, and so, by Remark 2.6(ii) and Lemma 6.24(ii), $e \circ g$, where e is an embedding of $\mathcal{G}$ into $\mathcal{F}$, is an embedding of $\mathcal{G}$ into $\mathcal{B}$ (in particular, is an isomorphism from $\mathcal{G}$ onto $\mathcal{B}$, for this has no proper submatrix). Thus, $|G| = |B| = |2| = 2$, in which case $G \supseteq 2$ is equal to 2, and so $2 = G$ forms a subalgebra of $\mathfrak{A}$, while $(\mathcal{A}{\restriction}2) = \mathcal{G}$ is canonically $\sim$-classical and isomorphic (and so equal) to $\mathcal{B}$. And what is more, by the truth-singularity of $\mathcal{A}$, $h(a') = 1$, for $(a'|1) \in D^{\mathcal{E}|\mathcal{A}}$, in which case $h(a) = h(\sim^{\mathfrak{E}} a') = \sim^{\mathfrak{A}} 1 = 0$, and so there is some $c \in (E \setminus \{a, a'\})$ such that $h(c) = \frac{1}{2}$. Then, $I \neq K \triangleq \{i \in I \mid \pi_i(c) = 1\} \neq \varnothing$, in which case $f \triangleq \{\langle\langle k, l\rangle, (K \times \{k\}) \cup ((I \setminus K) \times \{l\})\rangle \mid k, l \in 2\}$ is an embedding of $\mathcal{B}^2$ into $\mathcal{E}$, and so $(f \circ h) \in \mathrm{hom}(\mathcal{B}^2, \mathcal{A})$. Clearly, $f(\langle 1, 1\rangle) = a'$, $f(\langle 0, 0\rangle) = a$, $f(\langle 1, 0\rangle) = c$, and so $f(\langle 0, 1\rangle) = f(\sim^{\mathfrak{B}^2}\langle 1, 0\rangle) = \sim^{\mathfrak{E}} c$. Furthermore, the Σ-identity $\sim\sim x_0 \approx x_0$, being true in $\mathfrak{B}$, is so in $\mathfrak{A}$, for this belongs to the variety generated by $\mathfrak{B}$, in which case $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2} \notin 2$, and so $\sim^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2}$. Thus, $(f \circ h) = h_{+/2}$. Finally, if $\mathcal{A}$ was $\diamond$-conjunctive/-disjunctive, then, by Remark 2.8(ii)(a), (i)(a) and Lemma 6.18, it would be $\veebar$-disjunctive, where $\veebar \triangleq \diamond^{\sim/}$, for $\mathcal{B}$ would be so, in which case, by Theorem 3.9, $\mathcal{A}$ would be a strictly homomorphic counter-image of $\mathcal{B}$, and so, by Corollary 6.23(ii)⇒(iv), $\theta^{\mathcal{A}}$ would be a congruence of $\mathfrak{A}$. In this way, (2.14) and Corollary 6.23 complete the argument. □

In view of Example 1 of [18], this implies that U3VLSN are covered by the universal sequent approach elaborated therein and recently advanced in [24, 26] towards Hilbert-style axiomatizations. On the other hand, the item (ii) cannot be omitted in the formulation of Theorem 6.25, even if C is both weakly conjunctive and weakly disjunctive, in view of Remark 6.20 and:

Example 6.26. Let $\Sigma \triangleq \Sigma_{\sim,01}$ and $\mathcal{A}$ both truth-singular and involutive (in particular, non-$\sim$-negative) with $(\bot/\top)^{\mathfrak{A}} \triangleq (0/1)$. Then, $\mathcal{A}$ is both weakly $\bot$-conjunctive and weakly $\top$-disjunctive. Though, 2 forms a subalgebra of $\mathfrak{A}$, while $h_{+/2} \in \mathrm{hom}((\mathfrak{A}{\restriction}2)^2, \mathfrak{A})$, in which case, by Theorem 6.25, C is $\sim$-classical. □

Perhaps, a most remarkable peculiarity of non-classical U3VLSN is as follows.

6.2.2.1. Characteristic matrices.

Theorem 6.27. *Let $\mathcal{B}$ be a [canonical] $\sim$-super-classical Σ-matrix. Suppose C is non-$\sim$-classical and defined by $\mathcal{B}$. Then, $\mathcal{B}$ is isomorphic [and so equal] to $\mathcal{A}$. In particular, any uniform three-valued expansion of C is defined by a unique expansion of $\mathcal{A}$, unless C is $\sim$-classical.*

Proof. Then, the canonization $\mathcal{D}$ of $\mathcal{B}$ is isomorphic to $\mathcal{B}$, in which case, by (2.14), C is defined by $\mathcal{D}$, and so, by Theorem 6.25, both $\mathcal{A}$ and $\mathcal{D}$ are simple. Hence, by Remark 2.6(ii) and Lemma 3.7, $(\mathcal{A}|\mathcal{D}) \in \mathbf{H}(\mathbf{P}^{\mathrm{SD}}(\mathbf{S}(\mathcal{D}|\mathcal{A})))$ (in particular, $\mathcal{A}$ is truth-singular iff $\mathcal{D}$ is so, for truth-singularity is preserved under $\mathbf{P}$ as well as both $\mathbf{S}$ and $\mathbf{H}$; cf. Remark 2.8(ii)(c)). Therefore, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}(\mathcal{A})^I$, some subdirect product $\mathcal{E}$ of it and some $h \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{E},\mathcal{D})$, in which case, by (2.14) and Remark 2.8(ii)(b), $\mathcal{E}$ is a both consistent and truth-non-empty model of C, for $\mathcal{D}$ is so, and so $I \neq \varnothing$. Consider the following complementary cases:

- $(I\times\{j\}) \in E$, for some $j \in 2$, in which case $E \ni \sim^{\mathfrak{E}}(I\times\{j\}) = (I\times\{1-j\})$, and so, as $2 = \{j, 1-j\}$, E contains both of $(a|b) \triangleq (I \times \{1|0\})$. Consider the following complementary subcases:
  - $(I \times \{\frac{1}{2}\}) \in E$, in which case, as $I \neq \varnothing$, $g \triangleq \{\langle a', I \times \{a'\}\rangle \mid a' \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{E}$, and so, by Remark 2.6(ii), $g \circ h$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$ (in particular, is an isomorphism from $\mathcal{A}$ onto $\mathcal{D}$, because $|A| = 3 \leqslant l$, for no $l \in 3 = |D|$).
  - $(I \times \{\frac{1}{2}\}) \notin E$, in which case $\mathcal{E}$ is non-$\sim$-paraconsistent, and so is $\mathcal{B}$, in view of (2.14) (in particular, $\mathcal{A}$ is so). Then, 2 forms a subalgebra of $\mathfrak{A}$, for, otherwise, there would be some $\phi \in \mathrm{Fm}^2_\Sigma$ such that $\phi^{\mathfrak{A}}(1,0) = \frac{1}{2}$, in which case E would contain $\phi^{\mathfrak{E}}(a,b) = (I \times \{\frac{1}{2}\})$, and so, by (2.14), $\mathcal{F} \triangleq (\mathcal{A}\upharpoonright 2)$ is a canonical $\sim$-classical model of C (in particular, the logic C' of $\mathcal{F}$ is a $\sim$-classical extension of C). Moreover, as $a \in D^{\mathcal{E}} \not\ni b$, for $I \neq \varnothing$, $h(a) \in D^{\mathcal{D}} \not\ni h(b)$, in which case $h(b/a) = (0/1)$, whenever $\mathcal{D}$ is false-/truth-singular, respectively, and so $(1/0) = \sim^{\mathfrak{D}}(0/1) = h(\sim^{\mathfrak{E}}(b/a)) = h(a/b)$ (in particular, $h[\{a,b\}] = 2$). And what is more, as $h[E] = D$, there is some $c \in E$ such that $h(c) = \frac{1}{2}$. Let $\mathcal{G}$ be the submatrix of $\mathcal{E}$ generated by $\{a,b,c\}$, in which case $h' \triangleq (h\upharpoonright G) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{G},\mathcal{D})$, for $h[\{a,b,c\}] = A$, and so, by (2.14), C, being defined by $\mathcal{D}$, is defined by $\mathcal{G}$. Hence, $J \triangleq \{i \in I \mid \pi_i(c) = \frac{1}{2}\} \neq \varnothing$, for, otherwise, $2^I \supseteq \{a,b\}$ would contain c, in which case it, forming a subalgebra of $\mathfrak{A}^I$, would include G, and so $\mathcal{G}$, being a submatrix of $\mathcal{A}^I$, would be a submatrix of $\mathcal{F}^I \in \mathrm{Mod}(C')$ (in particular, by (2.14), C, being a sublogic of C', would be equal to C', and so would be $\sim$-classical, for C' is so). Take any $\jmath \in J \neq \varnothing$, in which case $\pi_\jmath(a|b|c) = (1|0|\frac{1}{2})$, and so $g' \triangleq (\pi_\jmath\upharpoonright G) \in \hom(\mathcal{G},\mathcal{A})$ is surjective, for $\{a,b,c\} \subseteq G$. We prove, by contradiction, that $g' \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{G},\mathcal{A})$. For suppose $g' \notin \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{G},\mathcal{A})$, in which case there is some $d \in (G \setminus D^{\mathcal{G}})$ such that $\pi_\jmath(d) \in D^{\mathcal{A}}$, and so $\pi_\jmath(\sim^{\mathfrak{G}} d) = \sim^{\mathfrak{A}}\pi_\jmath(d) \notin D^{\mathcal{A}}$, for, otherwise, (2.10) would not be true in $\mathcal{A}$ under $[x_0/\pi_\jmath(d), x_1/0]$. Then, $\sim^{\mathfrak{D}} d \notin D^{\mathcal{G}}$, in which case $\sim^{\mathfrak{D}} h'(d) = h'(\sim^{\mathfrak{G}} d) \notin D^{\mathcal{D}} \not\ni h'(d)$, and so $D^{\mathcal{D}} \not\ni h'(d) = \frac{1}{2}$ (in particular, $\mathcal{D}$ is truth-singular, that is, $\mathcal{A}$ is so). Let $\mathcal{H}$ be the submatrix of $\mathcal{G}$ generated by $\{a,b,d\}$, in which case $h'' \triangleq (h'\upharpoonright H) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{H},\mathcal{D})$, for $h'[\{a,b,d\}] = A$, since $h'(a|b|d) = (1|0|\frac{1}{2})$, respectively, and so, by (2.14), C, being defined by $\mathcal{D}$, is defined by $\mathcal{H}$. Then, as $\mathcal{A}$ is truth-singular, $\pi_\jmath(d) = 1$, in which case, for each $i \in J$, we get $\pi_i(d) = \pi_\jmath(d) = 1$, because $\pi_i(a|b|c) = (1|0|\frac{1}{2}) = \pi_\jmath(a|b|c)$, respectively, and so

$d \in 2^I \supseteq \{a, b\}$. Therefore, 2^I, forming a subalgebra of $\mathfrak{A}^I$, includes H, in which case $\mathcal{H}$, being a submatrix of $\mathcal{A}^I$, is that of $\mathcal{F}^I \in \mathrm{Mod}(C')$, and so, by (2.14), C, being a sublogic of C', is equal to C' (in particular, C is $\sim$-classical, for C' is so). This contradiction shows that $g' \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{G}, \mathcal{A})$. In this way, since both $\mathcal{A}$ and $\mathcal{D}$ are simple, while $h' \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{G}, \mathcal{D})$, by Remark 2.6(ii) and Lemma 3.6 with $\mathsf{M} = \{\mathcal{A}\}$, we eventually conclude that $\mathcal{A}$ is isomorphic to $\mathcal{D}$.

- $(I \times \{j\}) \notin E$, for each $j \in 2$, in which case, by Lemma 6.24(i), $\mathcal{A}$ is $\sim$-paraconsistent (in particular, false-singular, i.e., non-truth-singular), that is, $\mathcal{B}$ is so, and so $\mathcal{E}$ is $\sim$-paraconsistent, in view of (2.14), as well as is not truth-singular, in view Remark 2.8(ii)(c). Then, first, there is some $e \in D^{\mathcal{E}}$ such that $\sim^{\mathfrak{E}} e \in D^{\mathcal{E}}$, in which case $E \ni e \triangleq (I \times \{\frac{1}{2}\})$, and so $\mathcal{A}$ is extra-classically hereditary, for, otherwise, there would be some $\psi \in \mathrm{Fm}_{\Sigma}^{1}$ such that $j \triangleq \psi^{\mathfrak{A}}(\frac{1}{2}) \in 2$, in which case E would contain $\psi^{\mathfrak{E}}(e) = (I \times \{j\})$. Second, there is some $f \in D^{\mathcal{E}} \subseteq \{\frac{1}{2}, 1\}^I$ distinct from e, in which case $K \triangleq \{i \in I \mid \pi_i(f) = 1\} \neq \varnothing$, and so, since $\mathcal{A}$ is extra-classically hereditary and generated by $A \setminus \{0\}$, $g'' \triangleq \{\langle b', (K \times \{b'\}) \cup ((I \setminus K) \times \{\frac{1}{2}\})\rangle \mid b' \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{E}$. Hence, by Remark 2.6(ii), $g'' \circ h$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$, and so is an isomorphism from $\mathcal{A}$ onto $\mathcal{D}$, for $|A| = 3 = |D|$.

Thus, anyway, $\mathcal{A}$ is isomorphic to $\mathcal{D}$, and so to $\mathcal{B}$ [in which case, by Lemma 6.22(b), $\mathcal{A} = \mathcal{B}$]. Then, as $\sim$ is a subclassical negation for any expansion of C, (2.14) and Theorem 6.15 end the proof. □

In view of Theorem 6.27, $\mathcal{A}$, being uniquely determined by C, unless this is $\sim$-classical, is said to be *characteristic for/of C*. In view of Example 6.21, the stipulation of C's being non-$\sim$-classical cannot be omitted in the formulation of Theorem 6.27, even if C is both conjunctive and implicative (in particular, disjunctive).

Finally, Theorems 6.11(i)⇒(v) and 6.14 make the next paragraph equally acute.
6.2.2.2. Classical versus paraconsistent models and extensions.

Lemma 6.28. *Let $\mathcal{B}$ be a [classically hereditary {/weakly $\veebar$-disjunctive $\sim$-paraconsistent/($\veebar$, $\sim$)-paracomplete}] canonically $\sim$-[super-]classical Σ-matrix, C' the logic of $\mathcal{B}$ and $\mathcal{D}$ a consistent truth-non-empty [non-$\sim$-paraconsistent] (more specifically, $\sim$-classical; cf. Remark 2.8*(i)(c)*) model of C'. [Suppose either $\mathcal{D}$ is $\sim$-negative or $\mathcal{B}$ either is weakly conjunctive or is non-$\sim$-paraconsistent or has a ternary equalizer or has a quasi-negation.] Then, $\mathcal{B}[\upharpoonright 2]$ is is a canonical $\sim$-classical model of C', embeddable into a strictly surjectively homomorphic image of a submatrix of (and so isomorphic to) $\mathcal{D}$, in which case it defines a unique $\sim$-classical extension of C' [{in its turn, relatively axiomatized by* (3.2)/$(x_0 \veebar \sim x_0)$*}].*

Proof. We use Remark 2.8(i)(c/d) tacitly. Take any $a \in (D \setminus D^{\mathcal{D}}) \neq \varnothing$ and $b \in D^{\mathcal{D}} \neq \varnothing$. Then, by (2.14), the submatrix $\mathcal{E}$ of $\mathcal{D}$ generated by $\{a, b\}$ is a finitely-generated, consistent, truth-non-empty [non-$\sim$-paraconsistent] model of C' (equal to $\mathcal{D}$, for any $\sim$-classical Σ-matrix has no proper submatrix), an so, by Remarks 2.6(ii) and 2.8(ii)(b), $\mathcal{F} \triangleq (\mathcal{E}/\theta)$, where $\theta \triangleq \partial(\mathcal{E}) \in \mathrm{Con}(\mathcal{E})$, is a simple one ($\nu_\theta$ being an isomorphism from $\mathcal{E} = \mathcal{D}$ onto $\mathcal{F}$, for any $\sim$-classical Σ-matrix is simple). Hence, by Lemma 3.7, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{B}[\upharpoonright 2])^I$, some subdirect product $\mathcal{G}$ of it and some $h \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{G}, \mathcal{F})$, in which case, by [(2.14) and] Remark 2.8(ii)(b), $\mathcal{G}$ is both consistent and truth-non-empty [as well as non-$\sim$-paraconsistent]. Consider the following complementary cases:

- $\mathcal{B}$ is both $\sim$-paraconsistent and extra-classically hereditary, in which case it is involutive and classically hereditary, while $\{\frac{1}{2}, \sim^{\mathfrak{A}}\frac{1}{2}\} \subseteq D^{\mathcal{A}} \neq A$, and so $D^{\mathcal{A}} = \{\frac{1}{2}, 1\}$. Take any $c \in D^{\mathcal{G}} = (G \cap (D^{\mathcal{A}})^I) \neq \varnothing$ and $d \in (G \setminus D^{\mathcal{G}}) \neq \varnothing$, in which case $c \neq (I \times \{\frac{1}{2}\})$, for, otherwise, (2.10) would not be true in $\mathcal{G}$ under $[x_0/c; x_1/d]$, and so $J \triangleq \{i \in I \mid \pi_i(c) = 1\} \neq \varnothing$. Then, $\{\langle j, (I \times \{j\}) \cup ((I \setminus J) \times \{\frac{1}{2}\})\rangle \mid j \in 2\}$ is an embedding of $\mathcal{B}{\restriction}2$ into $\mathcal{G}$.

- $\mathcal{B}$ is not both $\sim$-paraconsistent and extra-classically hereditary, in which case, by Lemma 6.24(i,ii), $\mathcal{B}[{\restriction}2]$ is embeddable into $\mathcal{G}$.

Thus, anyway, there is some embedding e of $\mathcal{B}[{\restriction}2]$ into $\mathcal{G}$, in which case, as $\mathcal{B}[{\restriction}2]$, being $\sim$-classical, is simple, by Remark 2.6(ii), $f \triangleq (e \circ h)$ is am embedding of $\mathcal{B}[{\restriction}2]$ into $\mathcal{F}$ (and so $f \circ \nu_\theta^{-1}$ is an isomorphism from $\mathcal{B}[{\restriction}2]$ onto $\mathcal{D}$, for this, being $\sim$-classical, has no proper submatrix). In this way, (2.14) [{and "Theorem 3.12"/"Corollary 2.9 of [23]"}] end the proof. □

Lemma 6.29. *Suppose $\mathcal{A}$ is false-singular (in particular, $\sim$-paraconsistent) [while C is $\sim$-subclassical]. Then, the following are equivalent:*

(i) *C has no proper ([non-]non-$\sim$-subclassical) $\sim$-paraconsistent extension;*
(ii) *$\mathcal{A}$ either has a ternary {in particular, binary} semi-conjunction or is not extra-classically hereditary {in particular, not involutive};*
(iii) *$L_3 \triangleq (\Delta_2^- \cup \{\langle\frac{1}{2}, \frac{1}{2}\rangle\})$ does not form a subalgebra of $\mathfrak{A}^2$;*
(iv) *$\mathcal{A}_{\frac{1}{2}} \triangleq \langle\mathfrak{A}, \{\frac{1}{2}\}\rangle$ is not a $\sim$-paraconsistent model of C;*
(v) *C has no truth-singular $\sim$-paraconsistent model,*

in which case any three-valued expansion of C is maximally $\sim$-paraconsistent.

Proof. First, assume (ii) holds. Consider, any $\sim$-paraconsistent extension C' of C, in which case $x_1 \notin T \triangleq C'(\{x_0, \sim x_0\})$, and so, by the structurality of C', $\langle\mathfrak{Fm}^\omega_\Sigma, T\rangle \in \mathrm{Mod}(C')$. Then, by (2.14), $\mathcal{B} \triangleq \langle\mathfrak{Fm}^2_\Sigma, T \cap \mathrm{Fm}^2_\Sigma\rangle$ is a finitely-generated $\sim$-paraconsistent model of C' {in particular, of C}, in which case, by Lemma 3.7, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ and some subdirect product $\mathcal{D} \in \mathbf{H}^{-1}(\mathcal{B}/\partial(\mathcal{B}))$, and so $\mathcal{D}$ is a $\sim$-paraconsistent model of C'. Hence, there are some $a \in D^{\mathcal{D}}$ and some $b \in D$ such that $\sim^{\mathfrak{D}} a \in D^{\mathcal{D}} \not\ni b$, in which case $a = (I \times \{\frac{1}{2}\})$, while $I \supseteq J \triangleq \{i \in I \mid \pi_i(b) = 0\} \neq \varnothing$, whereas $\mathcal{D}$ is consistent. Consider the following complementary cases:

- $\mathcal{A}$ is extra-classically hereditary, in which case it is involutive. Then, $\mathcal{A}$ has a ternary semi-conjunction φ, in which case $c \triangleq \varphi^{\mathfrak{D}}(b, \sim^{\mathfrak{D}} b, a) \in (D \cap \{0, \frac{1}{2}\}^I)$, and so $\varnothing \neq J \subseteq K \triangleq \{i \in I \mid \pi_i(c) = 0\}$ {in particular, $\{\langle 0, c\rangle, \langle\frac{1}{2}, a\rangle, \langle 1, \sim^{\mathfrak{D}} c\rangle\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$}.
- $\mathcal{A}$ is not extra-classically hereditary, in which case there is some $\psi \in \mathrm{Fm}^1_\Sigma$ such that $j \triangleq \psi^{\mathfrak{A}}(\frac{1}{2}) \in 2$ {in particular, $D \ni d \triangleq \psi^{\mathfrak{D}}(a) = (I \times \{j\})$}. Therefore, as $2 = \{j, 1-j\}$, while $I \neq \varnothing$, $\{\langle j, d\rangle, \langle\frac{1}{2}, a\rangle, \langle 1-j, \sim^{\mathfrak{D}} d\rangle\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$.

Thus, anyway, $\mathcal{A}$ is embeddable into $\mathcal{D} \in \mathrm{Mod}(C')$, in which case, by (2.14), $\mathcal{A} \in \mathrm{Mod}(C')$, and $C' = C$. In this way, (i) holds.

Next, assume (iii) holds, while $\mathcal{A}$ is extra-classically hereditary, in which case it is involutive. Then, there is some $\phi \in \mathrm{Fm}^3_\Sigma$ such that $\bar{a} \triangleq \phi^{\mathfrak{A}^2}(\langle 0, 1\rangle, \langle 1, 0\rangle, \langle\frac{1}{2}, \frac{1}{2}\rangle) \notin L_3$, in which case $\{\frac{1}{2}\} \neq S \triangleq (\mathrm{img}\,\bar{a}) \neq 2$, and so $\varnothing \neq N \triangleq (S \cap 2) \subsetneq 2 \supseteq M \triangleq \{i \in 2 \mid a_i \in 2\} \neq \varnothing$ (in particular, N is a singleton). Let n be the unique element of $N \subseteq 2$, $m \triangleq \min(M) \in M \subseteq 2$ and $f : 2^2 \to 2, \langle j, k\rangle \mapsto |j - k|$, in which case $\sim^n(\phi[x_l/x_{f(m,l)}]_{l \in 2})$ is a ternary semi-conjunction for $\mathcal{A}$, and so (iii)⇒(ii) holds.

Conversely, assume (iii) does not hold, in which case, by (2.14), $\mathcal{E} \triangleq (\mathcal{A}^2{\restriction}L_3)$ is a model of C, and so is $\mathcal{A}_{\frac{1}{2}}$, for $(\pi_0{\restriction}L_3) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{E}, \mathcal{A}_{\frac{1}{2}})$, as $\mathcal{A}$ is false-singular. Then, $\mathcal{A}$ is extra-classically hereditary (in particular, involutive), for $L_3 \ni \langle\frac{1}{2}, \frac{1}{2}\rangle$ is disjoint with Δ_2, in which case $\mathcal{A}_{\frac{1}{2}}$ is $\sim$-paraconsistent, and so (iv) does not hold.

Furthermore, (iv) is a particular case of (v).

Further, assume (v) does not hold, that is, C has a truth-singular $\sim$-paraconsistent model $\mathcal{F}$. [Take any $\sim$-classical $\mathcal{G} \in \mathrm{Mod}(C)$.] Then, $x_0 \vdash {\sim}x_0$, not being true in any $\sim$-{super-}classical Σ-matrix, in view of (2.16) with $(m|n) = (1|0)$ {and (2.14) ⟨in particular, in $\mathcal{A}$⟩}, is true in $\mathcal{F}$, and so is its logical consequence $\{x_1, {\sim}x_1, x_0\} \vdash {\sim}x_0$ that, being a logical consequence of $(2.10)[x_k/{\sim}^k x_{1-k}]_{k\in 2}$, is true in any non-$\sim$-paraconsistent Σ-matrix [in particular, in $\mathcal{G}$; cf. Remark 2.8(i)(c)], in which case the logic of $\{\mathcal{F}[,\mathcal{G}]\}$ is a [non]-non-$\sim$-subclassical $\sim$-paraconsistent proper extension of C {in particular, (i) does not hold}.

Finally, Remark 2.8(i)(c), Theorem 6.27 and the fact that expansions of $\mathcal{A}$ retain ternary semi-conjunctions {if any} complete the argument. □

This {more precisely, its non-optional version} provides a purely-algebraic effective criterion of maximal paraconsistency of pararaconsistent U3VLSN.

Lemma 6.30. *Let $\mathcal{B}$ and $\mathcal{D}$ be Σ-matrices and $h \in \mathrm{hom}(\mathfrak{B}, \mathfrak{D})$. Suppose $\mathcal{B}$ is weakly $\veebar$-disjunctive, while $h[B] = D$, whereas $h[D^{\mathcal{B}}] = D^{\mathcal{D}}$. Then, $\mathcal{D}$ is weakly $\veebar$-disjunctive.*

Proof. Consider any $a \in D^{\mathcal{D}}$ and any $b \in D$. Then, there are some $c \in D^{\mathcal{B}}$ and some $d \in B$ such that $h(c|d) = (a|b)$, in which case $(c \veebar^{\mathfrak{B}} d) \in D^{\mathcal{B}} \ni (d \veebar^{\mathfrak{B}} c)$, for $\mathcal{B}$ is weakly $\veebar$-disjunctive, and so $\{a \veebar^{\mathfrak{D}} b, b \veebar^{\mathfrak{B}} a\} = h[\{c \veebar^{\mathfrak{B}} d, d \veebar^{\mathfrak{B}} c\}] \subseteq h[D^{\mathcal{B}}] = D^{\mathcal{D}}$. □

Corollary 6.31. *Suppose $\mathcal{A}$ is both $\sim$-paraconsistent and quadro-classically hereditary, while $\mathcal{B} \triangleq (\mathcal{A}^2 \restriction L_4)$ is $\veebar$-disjunctive. Then, $\mathcal{A}$ is classically hereditary.*

Proof. In that case, by (2.14), $\mathcal{B}$ is a $\sim$-negative model of C, while both $\pi_0[L_4] = A$ and $\pi_0[D^{\mathcal{B}}] = D^{\mathcal{A}}$, whereas $(\pi_0 \restriction L_4) \in \mathrm{hom}(\mathfrak{B}, \mathfrak{A})$, so, by Lemma 6.30, $\mathcal{A}$ is weakly $\veebar$-disjunctive. Then, by Remark 2.8(ii)(a),(i)(c) and Theorem 3.9, $\mathcal{A}$ has a $\sim$-negative (in particular, proper) submatrix $\mathcal{D}$, in which case it is both consistent and truth-non-empty, and so is not one-valued. On the other hand, the carriers of proper submatrices of $\mathcal{A}$ belong to $\{2, \{\frac{1}{2}\}\}$, in which case $D = 2$, and so $\mathcal{A}$ is classically hereditary. □

Theorem 6.32. *Let $\mathcal{B} \in \mathrm{Mod}(C)$ and C' the logic of $\mathcal{B}$. Suppose $\mathcal{B}$ is both truth-non-empty and consistent but not $\sim$-paraconsistent (more specifically, $\sim$-classical; cf. Remark 2.8*(i)(c)*), while either $\mathcal{A} \notin \mathrm{Mod}(C')$ or $\mathcal{B}$ is two-valued, whereas either $\mathcal{B}$ is $\sim$-negative or $\mathcal{A}$ either has a ternary equalizer or is weakly conjunctive or is non-$\sim$-paraconsistent or both has a quasi-negation and is either classically or quadro-classically hereditary. Then, there is some (non-proper) submatrix $\mathcal{D}$ of $\mathcal{B}$ such that the following hold:*

(i) *if $\mathcal{A}$ is classically hereditary, then $(\mathcal{A} \restriction 2) \in \mathrm{Mod}(C)$ is both canonically $\sim$-classical and embeddable into $\mathcal{D}/\eth(\mathcal{D})$ (and so isomorphic to $\mathcal{B}$, in which case it defines a unique $\sim$-classical extension of C);*

(ii) *if $\theta^{\mathcal{A}} \in \mathrm{Con}(\mathfrak{A})$ but $\mathcal{A}$ is not classically hereditary, then $\langle \chi^{\mathcal{A}}[\mathfrak{A}], \{1\}\rangle \in \mathrm{Mod}(C)$ is both canonically $\sim$-classical and embeddable into $\mathcal{D}/\eth(\mathcal{D})$ (and so isomorphic to $\mathcal{B}$, in which case it defines a unique $\sim$-classical extension of C);*

(iii) *if neither $\theta^{\mathcal{A}} \in \mathrm{Con}(\mathfrak{A})$ nor $\mathcal{A}$ is classically hereditary, then $\mathcal{A}$ is both $\sim$-paraconsistent and quadro-classically hereditary {in particular, involutive, while $\mathcal{B}$ is not disjunctive} but is neither weakly conjunctive nor extra-classically hereditary {in particular, C is maximally $\sim$-paraconsistent} as well as $\mathcal{A}^2 \restriction L_4$ is a strictly homomorphic counter-image of $\mathcal{D}/\eth(\mathcal{D})$ (whereas $\theta^{\mathcal{A}^2 \restriction L_4} \in \mathrm{Con}(\mathfrak{A}^2 \restriction L_4)$, in which case $\langle \chi^{\mathcal{A}^2 \restriction L_4}[\mathfrak{A}^2 \restriction L_4], \{1\}\rangle \in \mathrm{Mod}(C)$ is a canonically*

$\sim$-classical strictly surjectively homomorphic image of $\mathcal{A}^2{\restriction}L_4$ isomorphic to $\mathcal{B}$, and so defines a unique $\sim$-classical extension of C).

In particular, [providing C is not $\sim$-classical] C is [genuinely] $\sim$-subclassical if[f] $\mathcal{A}$ is [genuinely/conjunctively/disjunctively/implicatively] classically hereditary. Likewise, [providing C is not $\sim$-classical and (either disjunctive|"weakly conjunctive/implicative" or) non-$\sim$-paraconsistent] C is $\sim$-subclassical if[f] $\mathcal{A}$ is [(disjunctively|"weakly conjunctively/implicatively")] classically hereditary.

Proof. Take any $d \in D^{\mathcal{B}} \neq \varnothing$ and any $b \in (B \setminus D^{\mathcal{B}}) \neq \varnothing$. Then, by (2.14), the submatrix $\mathcal{D}$ of $\mathcal{B}$ generated by $\{b, d\}$ is a finitely-generated as well as non-$\sim$-paraconsistent (and equal to $\mathcal{B}$, for this has no proper submatrix) both consistent $\{$for $b \in D\}$ and truth-non-empty $\{$for $d \in D\}$ model of C' $\{$in particular, of $C\}$, and so $\mathcal{E} \triangleq (\mathcal{D}/\eth(\mathcal{D}))$ is a simple one, in view of Remarks 2.6(iv) and (2.8)(ii)(b) (while $\nu^{-1}_{\eth(\mathcal{B})}$ is an isomorphism from $\mathcal{E}$ onto $\mathcal{B}$, for this is simple). Assume both $\theta^{\mathcal{A}} \notin \mathrm{Con}(\mathfrak{A})$, in which case $\mathcal{A}$ is hereditarily simple, in view of Corollary 6.23(iii)$\Rightarrow$(iv), and $\mathcal{A}$ is not classically hereditary, in which case it is generated by 2, and so there is some $\varphi \in \mathrm{Fm}^2_\Sigma$ such that $\varphi^{\mathfrak{A}}(1,0) = \frac{1}{2}$. Then, by Lemma 3.7, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$, some subdirect product $\mathcal{F}$ of it and some $h \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{F}, \mathcal{E})$, in which case, by (2.14) and Remark 2.8(ii)(b), $\mathcal{F}$ is non-$\sim$-paraconsistent as well as both consistent and truth-non-empty, for $\mathcal{E}$ is so, and so is any Σ-matrix embeddable into $\mathcal{F}$. Therefore,

$$(I \times \{j\}) \notin F, \tag{6.3}$$

for all $j \in 2$, because, otherwise, by Lemma 6.24(ii), there would be some embedding e of $\mathcal{A}$, being generated by 2, into $\mathcal{F}$, in which case, by Remark 2.6(ii), $e \circ h$ would be an embedding of $\mathcal{A}$ into $\mathcal{E}$, and so, by (2.14), $\mathcal{A}$ would be a model of C', while $\mathcal{B}$ would not be two-valued, as $2 \not\geqslant 3 = |A| \leqslant |E| \leqslant |D| \leqslant |B|$. Hence, by (6.3) and Lemma 6.24(i), $\mathcal{A}$ is not weakly conjunctive but is $\sim$-paraconsistent, in which case $\{\frac{1}{2}, \sim^{\mathfrak{A}}\frac{1}{2}\} \subseteq D^{\mathcal{A}}$ $\{$in particular, $D^{\mathcal{A}} = \{\frac{1}{2}, 1\}\}$, and so

$$(I \times \{\tfrac{1}{2}\}) \notin F, \tag{6.4}$$

for $\mathcal{F}$ is consistent but not $\sim$-paraconsistent. Take any $a \in D^{\mathcal{F}} \neq \varnothing$, in which case $a \in \{\frac{1}{2}, 1\}^I$, and so, by (6.3) with $j = 1$ and (6.4), $I \neq J \triangleq \{i \in I \mid \pi_i(a) = 1\} \neq \varnothing$. Let $\mathcal{G}$ be the submatrix of $\mathcal{A}^2$ generated by $\{\langle 1, \frac{1}{2}\rangle\}$. Given any $x, y \in A$, set $(x \between y) \triangleq ((J \times \{x\}) \cup ((I \setminus J) \times \{y\})) \in A^I$, in which case $a = (1 \between \frac{1}{2})$, and so $g \triangleq \{\langle\langle x, y\rangle, (x \between y)\rangle \mid \langle x, y\rangle \in G\}$ is an embedding of $\mathcal{G}$ into $\mathcal{F}$ $\{$in particular, $(g \circ h) \in \hom(\mathcal{G}, \mathcal{E})$ is strict$\}$, for $I \neq J \neq \varnothing$. Then, by (6.3) and (6.4), G is disjoint with Δ_A. Let us prove, by contradiction, that G is disjoint with Δ^-_2. For suppose there is some $k \in 2$ such that $b \triangleq \langle k, 1-k\rangle \in G$, in which case, as $\{k, 1-k\} = 2 \ni 0 \notin D^{\mathcal{A}}$, $G \ni b \notin D^{\mathcal{G}} \not\ni \langle 1-k, k\rangle = \sim^{\mathcal{G}} b \in G$, while $\Delta^-_2 \subseteq G$, and so $\mathcal{G}$ is non-$\sim$-negative $\{$in particular, $\mathcal{B}$ is so, in view of Remark 2.8(ii)(a)$\}$, while, if $\mathcal{A}$ had a ternary equalizer τ, then G would contain $\tau^{\mathfrak{G}}(\langle 0,1\rangle, \langle 1,0\rangle, \langle 1, \frac{1}{2}\rangle) \in \Delta_A$, contrary to its being disjoint with Δ_A $\{$in particular, $\mathcal{A}$ is quadro-classically hereditary, for it is neither classically hereditary, nor weakly conjunctive nor non-$\sim$-paraconsistent$\}$. Hence, $L_4 \ni \langle 1, \frac{1}{2}\rangle$ includes G, in which case this is disjoint with Δ^-_2, for this is disjoint with L_4. This contradiction shows G is disjoint with Δ^-_2, in which case $G \subseteq L_4$, and so $\mathcal{A}$ is involutive, for, otherwise, as $\sim^{\mathfrak{A}}\frac{1}{2} \in D^{\mathcal{A}} = \{\frac{1}{2}, 1\}$, we would have $\sim^{\mathfrak{A}}\frac{1}{2} = 1$, in which case $G \ni \langle 1, \frac{1}{2}\rangle$ would contain $\sim^{\mathfrak{G}}\langle 1, \frac{1}{2}\rangle = \langle 0, 1\rangle \notin L_4$, and so would not be a subset of L_4. Let $\psi \triangleq (\varphi[x_1/\sim x_0]) \in \mathrm{Fm}^1_\Sigma$, in which case $\psi^{\mathfrak{A}}(1) = \varphi^{\mathfrak{A}}(1,0) = \frac{1}{2}$, and so $\psi^{\mathfrak{A}}(\frac{1}{2}) \in 2$ $\{$in particular, $\mathcal{A}$ is not extra-classically hereditary$\}$, for, otherwise, $G \ni \langle 1, \frac{1}{2}\rangle$ would contain $\psi^{\mathfrak{A}^2}(\langle 1, \frac{1}{2}\rangle) = \langle \frac{1}{2}, \frac{1}{2}\rangle \in \Delta_A$. Therefore, $\langle \frac{1}{2}, 1\rangle \in \{\psi^{\mathfrak{G}}(\langle 1, \frac{1}{2}\rangle), \sim^{\mathfrak{G}}\psi^{\mathfrak{G}}(\langle 1, \frac{1}{2}\rangle)\} \subseteq G$, in which case $G \supseteq \{\langle 1, \frac{1}{2}\rangle, \langle \frac{1}{2}, 1\rangle\}$ includes

$\sim^{\mathfrak{A}^2}[\{\langle 1, \frac{1}{2}\rangle, \langle\frac{1}{2}, 1\rangle\}] = \{\langle 0, \frac{1}{2}\rangle, \langle\frac{1}{2}, 0\rangle\}$, and so $G = L_4$ {in particular, $\mathcal{A}$ is quadro-classically hereditary, for G forms a subalgebra of $\mathfrak{A}^2$}. (Furthermore, $\chi^{\mathcal{B}}$, being injective, is an isomorphism from $\mathcal{B}$ onto $\mathcal{H} \triangleq \langle\chi^{\mathcal{B}}[\mathfrak{B}], \{1\}\rangle$, being thus canonically $\sim$-classical {in particular, simple}, in view of Remark 2.8(ii)(a), in which case $f \triangleq (((g \circ h) \circ \nu^{-1}_{\partial(\mathcal{B})}) \circ \chi^{\mathcal{B}}) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{G}, \mathcal{H})$ {in particular, $\mathfrak{H} = f[\mathfrak{G}]$}, for $\mathcal{H}$ has no proper submatrix, and so $\chi^{\mathcal{G}} = (f \circ \chi^{\mathcal{H}}) = (f \circ \Delta_2) = f$ {in particular, $\theta^{\mathcal{G}} = (\ker f) \in \mathrm{Con}(\mathfrak{G})$}. Then, by Remark 2.6(ii), f is injective.) Thus, (2.14), Remark 2.8(ii)(a), Corollaries 6.19, 6.31, Lemmas 6.18, 6.28, 6.29(ii)⇒(i) and Theorem 6.25 complete the argument. □

This {more precisely, its ()-optional version} provides an effective algebraic criterion of C's being $\sim$-subclassical, according to which C, being $\sim$-subclassical, has a unique /canonical $\sim$-classical extension/model to be denoted by $C^{\mathrm{PC}}/\mathcal{A}_{\mathrm{PC}}$ /"and constructed effectively from $\mathcal{A}$". Its item (ii) cannot be omitted, even if C is both conjunctive and implicative (and so disjunctive), in view of Example 6.21 and Corollary 6.23(iv)⇒(ii). Likewise, its item (iii) cannot be omitted, even if C is weakly disjunctive, in view of:

Example 6.33. Let $\Sigma \triangleq \{\vee, \sim\}$, $\mathcal{B}$ the canonically $\sim$-classical Σ-matrix with $(i \vee^{\mathfrak{B}} j) \triangleq 1$, for all $i, j \in 2$, and $\mathcal{A}$ both false-singular and involutive (in particular, $\sim$-paraconsistent) with $(a \vee^{\mathfrak{A}} b) \triangleq (\min(a, 1-a) + \frac{1}{2})$, for all $a, b \in A$, in which case, as $(\mathrm{img} \vee^{\mathfrak{A}/\mathfrak{B}}) \subseteq D^{\mathcal{A}/\mathcal{B}}$, $\mathcal{A}/\mathcal{B}$ is weakly $\vee$-disjunctive, and so $C/\mathcal{B}$ is weakly $\vee$-disjunctive/"$\vee^{\sim}$-conjunctive (cf. Remark 2.8(i)(a))", respectively. Then, we have $(\langle\frac{1}{2}|a, a|\frac{1}{2}\rangle \vee^{\mathfrak{A}^2} b) = \langle 1|\frac{1}{2}, \frac{1}{2}|1\rangle \in (L_4 \cap D^{\mathcal{A}^2})$, for all $a \in 2$ and all $b \in A^2$. Hence, $\mathcal{A}$ is quadro-classically hereditary, while $\chi^{\mathcal{A}^2 \upharpoonright L_4} \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{A}^2 \upharpoonright L_4, \mathcal{B})$, in which case, by (2.14), $\mathcal{B} \in \mathrm{Mod}(C)$, and so C is $\sim$-subclassical but is not $\sim$-classical in view of Remark 2.8(i)(c), whereas $(1 \vee^{\mathfrak{A}} 0) = (0 \vee^{\mathfrak{A}} 1) = \frac{1}{2}$, in which case $x_0 \vee x_1$ is a ternary equalizer for $\mathcal{A}$, as well as $\mathcal{A}$ is not classically hereditary, and so, by Theorems 6.25 and 6.32(iii), C is maximally $\sim$-paraconsistent but neither disjunctive nor weakly conjunctive nor genuinely $\sim$-subclassical (in particular, $\mathcal{B}$ is not genuinely $\sim$-classical, and so neither conjunctive nor disjunctive nor implicative). □

Finally, since $\mathcal{A}$ is weakly $\sim$-negative, whenever it is false-singular (in particular, $\sim$-paraconsistent), Remarks 2.4, 2.8(i)(a,c,d), Corollary 6.19, Theorems 6.25, 6.27, 6.32 and Lemma 6.29(ii)⇒(i) immediately yield:

Corollary 6.34. *$(x_0 \barwedge x_1)$ is a binary semi-conjunction of any [false-singular] canonical weakly $\barwedge$-conjunctive[(ly classically hereditary {in particular, genuinely/"weakly disjunctively ⟨more specifically, implicatively⟩" classically hereditary})] $\sim$-[super-]classical Σ-matrix [in which case its logic has no proper $\sim$-paraconsistent extension], and so C has no proper $\sim$-paraconsistent extension, whenever either of the following holds:*

(i) *$\mathcal{A}$ is not extra-classically hereditary (in particular, non-involutive {more specifically, classically-valued});*
(ii) *C is weakly conjunctive;*
(iii) *C is non-purely-inferential/"weakly disjunctive (in particular, implicative)" and is $\sim$-subclassical.*

This subsumes both the reference [Pyn95 b] in [15], going far beyond this, and all the $\sim$-paraconsistent inctances of U3VLSN summarized in Paragraph 6.2.1.1. Generally speaking, even $\sim$-subclassical maximally $\sim$-paraconsistent U3VLSN need not have theorems/"weakly conjunctive[ly classically hereditary] characteristic matrices", in view of Example 6.39 below /"as well as 6.33". On the other hand, the stipulation of C's being non-purely-inferential/$\sim$-subclassical cannot be omitted in Corollary 6.34(iii) /", even if C is strongly disjunctive", in view of the non-optional/optional version of:

Example 6.35. Let $\Sigma \triangleq (\Sigma_{\sim}[\cup\{\vee\}])$ and $\mathcal{A}$ both false-singular and involutive {and so $\sim$-paraconsistent, $\sim x_0$ being a quasi-negation for it} [with $\vee^{\mathfrak{A}} \triangleq ((\pi_0{\restriction}\Delta_A) \cup ((A^2 \setminus \Delta_A) \times \{\frac{1}{2}\}))$], in which case [$C$ satisfies (2.2) with $i = 0$ {for $\frac{1}{2} \in D^{\mathcal{A}}$} as well as both (2.3) and (2.4) {since both the commutativity and idempotence identities for $\vee$ are true in $\mathfrak{A}$}, and so, by Lemma 6.18, $C/\mathcal{A}$ is $\vee$-disjunctive, while $x_0 \vee x_1$ a ternary equalizer for $\mathcal{A}$, whereas] $\mathcal{A}$ is [neither] classically hereditary [nor quadro-classically hereditary, for both $(0 \vee^{\mathfrak{A}} 1) = \frac{1}{2} \notin 2$ and $(\langle\frac{1}{2}, 0\rangle \veebar^{\mathfrak{A}^2} \langle 0, \frac{1}{2}\rangle) = \langle\frac{1}{2}, \frac{1}{2}\rangle \notin L_4 \supseteq \{\langle\frac{1}{2}, 0\rangle, \langle 0, \frac{1}{2}\rangle\}$]. Then, L_3 forms a subalgebra of $\mathfrak{A}^2$, in which case, by Lemma 6.29(i/ii)$\Rightarrow$(iii), $C/\mathcal{A}$ has "a proper $\sim$-paraconsistent extension"/"no binary semi-conjunction", and so C is "[not] $\sim$-subclassical"/"not weakly conjunctive", in view of Theorem/Corollary 6.32/6.34. And what is more, in the non-optional case, Δ_2^- forms a subalgebra of $(\mathfrak{A}({\restriction}2))^2$, and so, by (2.14), $(\mathcal{A}({\restriction}2))^2{\restriction}\Delta_2^-$ is a truth-empty model of $C^{(\mathrm{PC})}$ (cf. Theorem 6.32(i)) {in particular, this has no theorem; cf. Corollary 3.10(ii)$\Rightarrow$(i)}. [On the other hand, $\mathcal{A}$, being false-singular and $\sim$-super-classical, is weakly $\sim$-negative, in which case it, being $\vee$-disjunctive, is not $(\vee, \sim)$-paracomplete, in view of Remark 2.8(i)(d), and so $x_0 \vee \sim x_0$ is a tautology/theorem of $\mathcal{A}/C$.] □

6.2.2.2.1. Maximal inferential consistency of non-subclassical non-paraconsistent U3VLSN.

Theorem 6.36. *Let $\mathcal{B}$ be a canonically $\sim$-[super-]classical Σ-matrix and C' the logic of $\mathcal{B}$. [Suppose $\mathcal{B}$ is either weakly conjunctive or non-$\sim$-paraconsistent.] Then, C' is maximally inferentially consistent [iff it is either $\sim$-classical or not $\sim$-subclassical], in which case it is "maximally consistent"/"structurally complete" iff it has theorems (i.e., the submatrix of $\mathcal{B}^{2[+1]}$ generated by $\{\langle 0, 1[, \frac{1}{2}]\rangle\}$ is truth-non-empty).*

Proof. [The "only if" part is by the inferential consistency of classical logics. Conversely, assume C' is not $\sim$-subclassical (the case, when C' is $\sim$-classical, is subsumed by the non-optional version), in which case, by (2.14), $\mathcal{A}$ is not classically hereditary, and so is generated by 2.] Consider any inferentially consistent extension C'' of C', in which case $x_1 \notin T \triangleq C''(x_0) \ni x_0$, while, by the structurality of C'', $\langle \mathfrak{Fm}_{\Sigma}^{\omega}, T\rangle$ is a model of C'' (in particular, of C'), and so is its finitely-generated consistent truth-non-empty submatrix $\mathcal{D} \triangleq \langle \mathfrak{Fm}_{\Sigma}^2, T \cap \mathrm{Fm}_{\Sigma}^2\rangle$, in view of (2.14). Then, by Lemma 3.7, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{B})^I$ and some subdirect product $\mathcal{E} \in \mathbf{H}^{-1}(\mathbf{H}(\mathcal{D}))$, in which case, by (2.14) and Remark 2.8(ii)(b), $\mathcal{E}$ is a consistent truth-non-empty model of C'', for $\mathcal{D}$ is so, and so, by Lemma 6.24(i,ii), $\mathcal{B}$ [being generated by 2] is embeddable into $\mathcal{E}$ (in particular, by (2.14), $C'' = C'$). In this way, Remark 2.9 and the inferential consistency of both C' and any consistent logic with theorems complete the argument. □

This (more precisely, its non-optional version) subsumes both a quite effective algebraic criterion of structural completeness of classical logics and the maximal {inferential} consistency of classical logics with{out} theorems. In the {after} next subparagraph, we study the relative one of unique classical extensions of subclassical U3VLSN.

6.2.2.2.2. Relative maximal consistency of classical extensions versus theorems and binary semi-conjunctions.

Lemma 6.37. *Let $\mathcal{B}$ be a canonically $\sim$-[super-]classical Σ-matrix and φ a binary semi-conjunction for it. [Suppose $\mathcal{B}$ is either false-singular or not simple or both classically hereditary and not extra-classically hereditary.] Then, it has a tautology.*

Proof. Then, $\phi \triangleq \sim(\varphi[x_1/\sim x_0]) \in \mathrm{Fm}_\Sigma^1$ is a tautology of $\mathcal{B}$, whenever this is two-valued. [Otherwise, consider the following exhaustive cases:

- $\mathcal{B}$ is false-singular, in which case $D^{\mathcal{B}} = \{\frac{1}{2}, 1\}$, and so $\phi(\phi)$ is a tautology of $\mathcal{B}$.
- $\mathcal{B}$ is not simple, in which case, by Corollary 6.23(iii)⇒(iv), $\theta^{\mathcal{B}} \in \mathrm{Con}(\mathfrak{B})$, while $h \triangleq \chi^{\mathcal{B}}$ is a strict surjective homomorphism from $\mathcal{B}$ onto the canonically $\sim$-classical Σ-matrix $\mathcal{D} \triangleq \langle h[\mathfrak{B}], \{1\}\rangle$, and so, by (2.14), $\mathcal{D}$ and $\mathcal{B}$ define the same logic. Then, $h{\restriction}2$ is diagonal, in which case φ is a binary semi-conjunction for $\mathcal{D}$, and so ϕ is a tautology of $\mathcal{D}$ (in particular, of $\mathcal{B}$).
- $\mathcal{B}$ is both classically hereditary and not extra-classically hereditary, in which case there is some $\psi \in \mathrm{Fm}_\Sigma^1$ such that $\psi^{\mathfrak{B}}(\frac{1}{2}) \in 2$, and so $\phi(\psi)$ is a tautology of $\mathcal{B}$.] □

Theorem 6.38. *Suppose C is $\sim$-suclassical, while $\mathcal{A}$ is [not] false-singular. Then, the following are equivalent:*

(i) *$\mathcal{A}$ has a tautology (in particular, is weakly disjunctive [but is not extra-classically hereditary]);*
(ii) *C^{PC} has a theorem [while $\mathcal{A}$ is not extra-classically hereditary];*
(iii) *$\mathcal{A}$ has a binary semi-conjunction (in particular, is weakly conjunctive{ly/disjunctively classically hereditary}) [but is not extra-classically hereditary];*
(iv) *C^{PC} is C-relatively maximally consistent.*

Proof. First, (i)⇒(ii) is by the inclusion $C(\varnothing) \subseteq C^{\mathrm{PC}}(\varnothing)$ [and Corollary 3.10(iv)⇒(i)], while (iii)⇒(i) is by [Theorems 6.25, 6.32(iii) and Corollary 6.23(iv)⇒(iii) as well as] Lemma 6.37, whereas (iv)⇒(i) is by Remark 2.9.

Next, assume C^{PC} has a theorem, in which case, by Remark 2.4, there is some $\varphi \in (\mathrm{Fm}_\Sigma^1 \cap C^{\mathrm{PC}}(\varnothing))$, and so this is a tautology of $\mathcal{A}_{\mathrm{PC}}$. Consider the following complementary cases:

- $\mathcal{A}$ is classically hereditary, in which case, by Theorem 6.32(i), $\mathcal{A}_{\mathrm{PC}} = (\mathcal{A}{\restriction}2)$, and so $\sim\varphi$ is a semi-conjunction for $\mathcal{A}{\restriction}2$ (in particular, a binary one for $\mathcal{A}$).
- $\mathcal{A}$ is not classically hereditary. Consider the following complementary subcases:
  - $\theta^{\mathcal{A}} \in \mathrm{Con}(\mathcal{A})$, in which case, by Theorem 6.32(ii), $\mathcal{A}_{\mathrm{PC}} = \mathcal{B} \triangleq \langle \chi^{\mathcal{A}}[\mathfrak{A}], \{1\}\rangle$, and so $h \triangleq \chi^{\mathcal{A}} \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{A}, \mathcal{B})$. Consider the following complementary subsubcases:
    * $\mathcal{A}$ is truth-singular, in which case, by (2.14), φ is a tautology of $\mathcal{A}$, and so $\sim\varphi$ is a binary semi-conjunction for $\mathcal{A}$.
    * $\mathcal{A}$ is false-singular, in which case, for each $i \in 2$, $\sim^{\mathfrak{B}}\varphi^{\mathfrak{B}}(i) = 0 \notin D^{\mathcal{B}}$, and so $\sim^{\mathfrak{A}}\varphi^{\mathfrak{A}}(i) = 0$, for $h{\restriction}2$ is diagonal (in particular, $\sim\varphi$ is a binary semi-conjunction for $\mathcal{A}$).
  - $\theta^{\mathcal{A}} \notin \mathrm{Con}(\mathcal{A})$, in which case, by Theorem 6.32(iii), $\mathcal{A}$ is $\sim$-paraconsistent (in particular, false-singular) and quadro-classically hereditary, while $\mathcal{D} \triangleq (\mathcal{A}^2{\restriction}L_4)$ is a strictly surjectively homomorphic counter-image of $\mathcal{A}_{\mathrm{PC}}$, and so, by (2.14), φ is a tautology of $\mathcal{D}$. Then, for each $i \in 2$, $\varphi^{\mathfrak{D}}(\langle \frac{1}{2}, i\rangle) \in D^{\mathcal{D}} = \{\langle \frac{1}{2}, 1\rangle, \langle 1, \frac{1}{2}\rangle\}$. Consider the following complementary subsubcases:
    * $\varphi^{\mathfrak{A}}(\frac{1}{2}) = \frac{1}{2}$, in which case $\varphi^{\mathfrak{D}}(\langle \frac{1}{2}, i\rangle) = \langle \frac{1}{2}, 1\rangle$, and so $\sim\varphi$ is a binary semi-conjunction for $\mathcal{A}$.
    * $\varphi^{\mathfrak{A}}(\frac{1}{2}) \neq \frac{1}{2}$, in which case $\varphi^{\mathfrak{D}}(\langle \frac{1}{2}, i\rangle) = \langle 1, \frac{1}{2}\rangle$, and so $\sim\varphi(\varphi)$ is a binary semi-conjunction for $\mathcal{A}$.

Thus, (ii)⇒(iii) holds.

Further, assume (iii) holds, in which case (i) does so, and so $C(\varnothing) \neq \varnothing$. Consider any consistent extension C' of C. If $C' = C$, then it is clearly a sublogic of C^{PC}. Now, assume $C' \neq C$, in which case, by (iii) and Lemma 6.29(iii)⇒(i), C' is non-$\sim$-paraconsistent, and so is any model of it. Then, $x_0 \notin T \triangleq C'(\varnothing) \supseteq C(\varnothing) \neq \varnothing$, in which case, by the structurality of C', $\mathcal{D} \triangleq \langle \mathfrak{Fm}^{\omega}_{\Sigma}, T\rangle$ is a consistent truth-non-empty model of C' (in particular, $\mathcal{A}$ is not a model of the logic of $\mathcal{D}$, for C' is a proper extension of C), and so a non-$\sim$-paraconsistent one of C. Hence, by (iii), (2.14) and Theorem 6.32, $\mathcal{A}_{\mathrm{PC}}$ is a model of C', that is, C^{PC} is an extension of C'. Thus, (iv) holds.

Finally, Remark 2.8(i)(a,d),(ii)(a), Corollaries 3.10(i)⇒(iv), 6.34 and the weak $\sim$-negativilty of false-singular $\sim$-super-classical Σ-matrices complete the proof. □

This provides an effective algebraic algebraic criterion of the relative maximal consistency of unique $\sim$-classical extensions of $\sim$-subclassical uniformly three-valued Σ-logics with subclassical negation $\sim$, the non-triviality of the property involved being due to:

Example 6.39. Let $\Sigma = \Sigma^{-}_{\sim}$ with unary $\neg$ and $\mathcal{A}$ both truth-/false-singular, non-$\sim$-negative /(and so $\sim$-paraconsistent) and non-invlolutive with $\neg^{\mathfrak{A}} a \triangleq (1 - a)$, for all $a \in A$, in which case it is classically hereditary but not extra-classically hereditary, while $\neg x_0$ is a a quasi-negation for it, and so by Theorem 6.32(i) /"and Corollary 6.34(i)", C is $\sim$-subclassical /"and maximally $\sim$-paraconsistent". Nevertheless, Δ^{-}_{2} forms a subalgebra of $(\mathfrak{A}[\restriction 2])^2$, in which case, by (2.14), $(\mathcal{A}[\restriction 2])^2 \restriction \Delta^{-}_{2}$ is a truth-non-empty model of $C^{[\mathrm{PC}]}$ [cf. Theorem 6.32(i)], and so this has no theorem (in particular, by Lemma 6.37, $\mathcal{A}[\restriction 2]$ has no binary semi-conjunction, and so, by Corollary 6.34, $\mathcal{A}\restriction 2$ is not weakly conjunctive /"as well as $\mathcal{A}$ is not weakly conjunctive{ly classically hereditary}"). □

6.2.2.2.3. Relative maximal inferential consistency of classical extensions versus proper paraconsistent extensions and quasi-negations.

Theorem 6.40. *Suppose C is $\sim$-subclassical and does [not] satisfy* (2.10). *Then, C^{PC} is C-relatively maximally inferentially consistent [iff the following hold:*

(i) *C has no proper {more specifically, non-$\sim$-subclassical} $\sim$-paraconsistent extension (i.e., L_3 does not form a subalgebra of $\mathfrak{A}^2$; cf. Lemma 6.29(i)⇔(iii));*
(ii) *$\mathcal{A}$ has a quasi-negation (in particular, is involutive)].*

Proof. [First, assume C^{PC} is C-relatively maximally inferentially consistent, in which case (i) is by the inferential consistency of $\sim$-paraconsistent Σ-logics. Furthermore, we prove (ii), by contradiction. For suppose (ii) does not hold, in which case $\mathcal{A}$ is not involutive, and so $\sim^{\mathfrak{A}}\frac{1}{2} = 1$, for $\mathcal{A}$ is $\sim$-paraconsistent. Let $\mathcal{B}$ be the submatrix of $\mathcal{A}^2$ generated by $S \triangleq \{\langle 1, \frac{1}{2}\rangle\}$, in which case $B \supseteq (S \cup \sim^{\mathfrak{A}^2}[S] \cup \sim^{\mathfrak{A}^2}[\sim^{\mathfrak{A}^2}[S]]) = (S \cup \Delta^{-}_{2})$ is disjoint with $\{0, \frac{1}{2}\}^2$, and so $D^{\mathcal{B}} \supseteq S \neq \varnothing$, while $(B \setminus D^{\mathcal{B}}) = \Delta_2 \neq \varnothing$ (in particular, $\mathcal{B}$ is both consistent and truth-non-empty). Then, by (2.14), the logic C' of $\mathcal{B}$ is an inferentially consistent extension of C, in which case C' is a sublogic of C^{PC}. On the other hand, $\sim^{\mathfrak{A}^2}[\Delta^{-}_{2}] \subseteq \Delta^{-}_{2}$, in which case the Σ-rule $\sim x_0 \vdash x_0$, not being true in $\mathcal{A}_{\mathrm{PC}}$ under $[x_0/0]$, is true in $\mathcal{B}$, and so this contradiction shows that (ii) holds. Conversely, assume (i,ii) hold.] Consider any inferentially consistent extension C'' of C, in which case it is sublogic of the extension C^{PC} of C, whenever $C'' = C$. Now, assume $C'' \neq C$, in which case C'' is not $\sim$-paraconsistent, while, by Theorem 6.36, C is not $\sim$-classical, and so, by Theorems 6.25 and 6.32, $\mathcal{A}$ is ether classically or quadro-classically hereditary. Then, $x_1 \notin T \triangleq C''(x_0) \ni x_0$, in which case, by the structurality of C'', $\mathcal{D} \triangleq \langle \mathfrak{Fm}^{\omega}_{\Sigma}, T\rangle$ is a consistent truth-non-empty model of C'', and so a non-$\sim$-paraconsistent one of C. Hence, $\mathcal{A}$ is not a model of the logic C''' of $\mathcal{D} \in \mathrm{Mod}(C'')$, for C'' is a proper extension of C. And what is more, $\mathcal{A}$ has a quasi-negation, whenever it is $\sim$-paraconsistent. Then, by (2.14) and Theorem 6.32, $\mathcal{A}_{\mathrm{PC}}$ is a model of C''', so C'' is a sublogic of C^{PC}. □

This provides an effective algebraic algebraic criterion of the relative maximal inferential consistency of unique $\sim$-classical extensions of $\sim$-subclassical uniformly three-valued Σ-logics with subclassical negation $\sim$, the necessity of its item (i/ii) (and so the non-triviality of the property involved) in the "$\sim$-paraconsistent" case being due to /"the optional version of" Example 6.35/6.44 /below. Though inferentially consistent logics are consistent, Theorem 6.40 is not subsumed by Theorem 6.38, in view of Example 6.39.

6.2.2.2.4. Structural completeness of paraconsistent U3VLSN versus maximal paraconsistency and ternary equalizers.

Lemma 6.41. *Let ϵ be a ternary equalizer for $\mathcal{A}$. Suppose $\mathcal{A}$ is false-singular but is not classically-hereditary. Then, it has a tautology.*

Proof. In that case, there is some $\phi \in \mathrm{Fm}^2_\Sigma$ such that $\phi^{\mathfrak{A}}(0,1) = \frac{1}{2}$. Then, $\psi \triangleq (\phi[x_1/{\sim}x_0]) \in \mathrm{Fm}^1_\Sigma$, while $\psi^{\mathfrak{A}}(0) = \phi^{\mathfrak{A}}(0,1) = \frac{1}{2}$. Consider the following complementary cases:

- $(\psi^{\mathfrak{A}}[D^{\mathcal{A}}] \cap \{1\}) = \varnothing$, in which case ${\sim}\psi$ is a tautology of $\mathcal{A}$.
- $(\psi^{\mathfrak{A}}[D^{\mathcal{A}}] \cap \{1\}) \neq \varnothing$. Consider the following complementary subcases:
  - $\psi^{\mathfrak{A}}[D^{\mathcal{A}}] = \{1\}$, in which case $\psi(\psi)$ is a tautology of $\mathcal{A}$.
  - $\psi^{\mathfrak{A}}[D^{\mathcal{A}}] \neq \{1\}$. Consider the following complementary subcases:
    * $(\psi^{\mathfrak{A}}[D^{\mathcal{A}}] \cap \{0\}) = \varnothing$, in which case ψ is a tautology of $\mathcal{A}$.
    * $(\psi^{\mathfrak{A}}[D^{\mathcal{A}}] \cap \{0\}) \neq \varnothing$, in which case there are some $a, b \in D^{\mathcal{A}}$ such that $\psi^{\mathfrak{A}}(a) = 0$ and $\psi^{\mathfrak{A}}(b) = 1$, in which case $a \neq b$, and so $\{a, b\} = D^{\mathcal{A}}$. Put $\varphi \triangleq (\psi[x_{(2\cdot b)-1}/\psi({\sim}x_0)]) \in \mathrm{Fm}^1_\Sigma$. Then, $\varphi^{\mathfrak{A}}(1) = 1$ and $\varphi^{\mathfrak{A}}(0) = \frac{1}{2}$. Set $\xi \triangleq \epsilon({\sim}x_0, x_0, \varphi) \in \mathrm{Fm}^1_\Sigma$. Then, $\xi^{\mathfrak{A}}(0) = \xi^{\mathfrak{A}}(1)$. Let $i \triangleq (1 - \chi^{\mathcal{A}}(\xi^{\mathfrak{A}}(0))) \in 2$ and $\eta \triangleq {\sim}^i\xi \in \mathrm{Fm}^1_\Sigma$. Then, $\eta^{\mathfrak{A}}[2] \subseteq D^{\mathcal{A}}$, while $\eta^{\mathfrak{A}}(0) = \eta^{\mathfrak{A}}(1)$. Let $j \triangleq \chi^{\mathcal{A}}(\eta^{\mathfrak{A}}(\frac{1}{2})) \in 2$ and $k \triangleq \max(j, 1 - \chi^2_A(\eta^{\mathfrak{A}}(0))) \in 2$. Then, ${\sim}^{(1-j)\cdot k}(\eta[x_k/\eta]) \in \mathrm{Fm}^1_\Sigma$ is a tautology of $\mathcal{A}$. □

Lemma 6.42. *Suppose $\mathcal{A}$ [both] is classically hereditary (in which case C is $\sim$-subclassical with $\mathcal{A}_{\mathrm{PC}} = (\mathcal{A}{\restriction}2)$; cf. Theorem 6.32(i)) [and either is false-singular or has a ternary anti-equalizer, as well as C is not axiomatically equivalent to C^{PC}]. Then, the logic C' of $\mathcal{B} \triangleq (\mathcal{A} \times \mathcal{A}_{\mathrm{PC}})$ is a [proper] axiomatically-equivalent extension of C.*

Proof. Clearly, $(\pi_0{\restriction}B) \in \hom(\mathcal{B}, \mathcal{A})$ is surjective, in which case, by (2.14) and (2.15), C' is an axiomatically-equivalent extension of C. [Take any $\varphi \in (C^{\mathrm{PC}}(\varnothing) \setminus C(\varnothing)) \neq \varnothing$, for $C^{\mathrm{PC}}(\varnothing) \nsubseteq C(\varnothing)$, as $C(\varnothing) \subsetneq C^{\mathrm{PC}}(\varnothing)$, in which case φ is true in $\mathcal{A}{\restriction}2$, while there is some $h \in \hom(\mathrm{Fm}^\omega_\Sigma, \mathfrak{A})$ such that $h(\varphi) \notin D^{\mathcal{A}}$, whereas $V \triangleq \mathrm{Var}(\varphi) \subseteq \mathrm{Var}_\omega$ is finite, and so $|\mathrm{Var}_\omega \setminus V| = |\mathrm{Var}_\omega| = \omega \supseteq 2$, for ω is infinite. Take any injective $\bar{v} : 2 \to (\mathrm{Var}_\omega \setminus V)$. Consider the following complementary cases:

- $h(\varphi) = 0$, in which case $h({\sim}\varphi) = 1 \in D^{\mathcal{A}}$, while, for all $g \in \hom(\mathrm{Fm}^\omega_\Sigma, \mathfrak{A}{\restriction}2)$, $g(\varphi) = 1$, and so $g({\sim}\varphi) = 0 \notin D^{\mathcal{A}}$. Then, ${\sim}\varphi \vdash v_0$ is true in $\mathcal{B}$ but is not true in $\mathcal{A}$ under $(h{\restriction}(\mathrm{Var}_\omega \setminus \{v_0\})) \cup [v_0/0]$, for $0 \notin D^{\mathcal{A}}$.
- $h(\varphi) \neq 0$, in which case $\mathcal{A}$ is not false-singular, that is, truth-singular, and so has a ternary anti-equalizer τ, while $h(\varphi) = \frac{1}{2}$, for $1 \in D^{\mathcal{A}}$. Let $k \triangleq \tau^{\mathfrak{A}}(0,1,1) \in 2$ and $\theta \triangleq {\sim}^k\tau$, in which case $\theta^{\mathfrak{A}^2}(\langle 0,0\rangle, \langle 1,1\rangle, \langle \frac{1}{2}, 1\rangle) = \langle 1, 0\rangle$, and so $\{v_0, \theta({\sim}v_0, v_0, \varphi)\} \vdash v_1$ is true in $\mathcal{B}$, for $\theta^{\mathfrak{A}}(0,1,1) = 0 \notin$

$D^{\mathcal{A}} = \{1\}$, but is not true in $\mathcal{A}$ under $(h{\restriction}(\mathrm{Var}_\omega \setminus \{v_0, v_1\})) \cup [v_0/1, v_1/0]$, for $\theta^{\mathfrak{A}}(0, 1, \frac{1}{2}) = 1 \in D^{\mathcal{A}} \not\ni 0$.] □

Theorem 6.43. *Suppose $\mathcal{A}$ is false-singular (in particular, $\sim$-paraconsistent). In that case, C is structurally complete iff the following hold:*

(i) *C has a theorem (i.e., the submatrix of $\mathcal{A}^3$ generated by $\{\langle 0, 1, \frac{1}{2}\rangle\}$ is truth-non-empty), whenever it is $\sim$-classical (i.e., $\theta^{\mathcal{A}} \in \mathrm{Con}(\mathfrak{A})$; cf. Theorem 6.25);*

(ii) *C has no proper $\sim$-paraconsistent extension (i.e., L_3 does not form a subalgebra of $\mathfrak{A}^2$, that is, either $\mathcal{A}$ is not extra-classically hereditary [in particular, is non-involutive {more specifically, is classically-valued}] or has a ternary ⟨in particular, binary⟩ semi-conjunction; cf. Lemma 6.29(i)⇔(ii)⇔(iii));*

(iii) *$\mathcal{A}$ is not classically hereditary, whenever C is not $\sim$-classical (i.e., $\theta^{\mathcal{A}} \notin \mathrm{Con}(\mathfrak{A})$; cf. Theorem 6.25);*

(iv) *$\mathcal{A}$ is not quadro-classically hereditary (in particular, non-involutive);*

(v) *$\mathcal{A}$ has a ternary equalizer (in particular, either has a binary semi-conjunction or is $\sim$-negative).*

In particular, providing C is weakly conjunctive, it is structurally complete iff it is either $\sim$-classical or non-$\sim$-subclassical.

Proof. First, assume (i–v) hold, in which case by (i,iii,v) and Lemma 6.41, C has a theorem, and so is structurally complete, unless it is $\sim$-paraconsistent, in view of Theorems 6.25, 6.32, 6.36 and (iii). Now, assume C is $\sim$-paraconsistent, in which case, by Remark 2.8(i)(c), C is not $\sim$-classical, and so, by Theorem 6.25, $\theta^{\mathcal{A}} \notin \mathrm{Con}(\mathfrak{A})$, while, by (iii), $\mathcal{A}$ is not classically hereditary. Consider any axiomatically-equivalent extension C' of C, in which case it is consistent, for C is so, as $\mathcal{A}$ is so, and so $x_1 \notin T \triangleq C'(\varnothing) \supseteq C(\varnothing) \neq \varnothing$. Then, by the structurality of C', $\mathcal{B} \triangleq \langle \mathfrak{Fm}^{\omega}_{\Sigma}, T\rangle$ is a consistent truth-non-empty model of C' (in particular, of its sublogic C). We prove that $C' = C$, by contradiction. For suppose $C' \neq C$, in which case, by (ii), C' is non-$\sim$-paraconsistent, and so is $\mathcal{B} \in \mathrm{Mod}(C')$. Then, by (iv,v) and Theorem 6.32(iii), $\mathcal{A}$ is a model of the logic of $\mathcal{B} \in \mathrm{Mod}(C')$, and so of C', in which case $C' = C$ (in particular, C is structurally complete). Conversely, assume either of (i–iv) does not hold. Consider the respective cases:

(i) does not hold, in which case C is purely-inferential as well as inferentially consistent, for $\mathcal{A}$ is both consistent and truth-non-empty, and so, by Remark 2.9, C is not structurally complete.

(ii) does not hold, in which case $\mathcal{A}$ is extra-classically hereditary, for $L_3 \ni \langle \frac{1}{2}, \frac{1}{2}\rangle$ is disjoint with Δ_2, and so involutive, while, by (2.14), $\mathcal{D} \triangleq (\mathcal{A}^2{\restriction}L_3) \in \mathrm{Mod}(C)$, and so, by (2.15), the logic C'' of $\mathcal{D}$ is an axiomatically-equivalent extension of C, for $(\pi_0{\restriction}D) \in \hom(\mathcal{D}, \mathcal{A})$ is surjective, as $\pi_0[L_3] = A$. Then, $D^{\mathcal{D}} = \{\langle \frac{1}{2}, \frac{1}{2}\rangle\}$, in which case, the Σ-rule $x_0 \vdash \sim x_0$, not being true in $\mathcal{A}$ under $[x_0/1]$, is true in $\mathcal{D}$, for $\mathcal{A}$ is involutive, and so C'' is a proper extension of C (in particular, C is not structurally complete).

(iii) does not hold, in which case both C is non-$\sim$-classical and $\mathcal{A}$ is classically hereditary, in which case, by Theorem 6.32(i), C is $\sim$-subclassical (in particular, is a proper sublogic of C^{PC}), and so is not structurally complete, whenever it is axiomatically-equivalent to C^{PC}. Otherwise, it is not structurally complete as well, in view of Lemma 6.42.

(iv) does not hold, in which case $\mathcal{A}$ is involutive (in particular, is $\sim$-paraconsistent, for it is false-singular), for $L_4 \ni \langle 0, \frac{1}{2}\rangle$ is disjoint with 2^2, while, by (2.14), $\mathcal{F} \triangleq (\mathcal{A}^2{\restriction}L_4) \in \mathrm{Mod}(C)$ is not $\sim$-paraconsistent, for $\langle \frac{1}{2}, \frac{1}{2}\rangle \notin L_4$. Then, by

(2.15), the logic of $\mathcal{F}$ is a non-$\sim$-paraconsistent (and so proper) axiomatically-equivalent extension of C, for $(\pi_0 \restriction F) \in \hom(\mathcal{F}, \mathcal{A})$ is surjective, as $\pi_0[L_3] = A$, and so C is not structurally complete.

(v) does not hold, in which case $\sim^{\mathfrak{A}} \frac{1}{2} \neq 0$ (in particular, $\mathcal{A}$, being false-singular, is $\sim$-paraconsistent), for, otherwise, $\sim x_2$ would be a ternary equalizer for $\mathcal{A}$, while, by (2.14), the submatrix $\mathcal{G}$ of $\mathcal{A}^2$ generated by $M \triangleq (\Delta_2^- \cup \{\langle \frac{1}{2}, 1\rangle\})$ is a model of C, whereas G is disjoint with $\Delta_A \ni \langle \frac{1}{2}, \frac{1}{2}\rangle$ (in particular, $\mathcal{G}$ is not $\sim$-paraconsistent), and so, by (2.15), the logic of $\mathcal{G}$ is a non-$\sim$-paraconsistent (and so proper) axiomatically-equivalent extension of C (in particular, C is not structurally complete), for $(\pi_0 \restriction G) \in \hom(\mathcal{G}, \mathcal{A})$ is surjective, as $\pi_0[M] = A$.

Finally, assume C is weakly $\bar{\wedge}$-conjunctive, in which case $(\frac{1}{2} \bar{\wedge}^{\mathfrak{A}} 0) = 0 = (0 \bar{\wedge}^{\mathfrak{A}} \frac{1}{2})$, for $\mathcal{A}$ is false-singular with non-distinguished value 0, and so $(\langle \frac{1}{2}, 0\rangle \bar{\wedge}^{\mathfrak{A}^2} \langle 0, \frac{1}{2}\rangle) = \langle 0, 0\rangle \notin L_4 \supseteq \{\langle \frac{1}{2}, 0\rangle, \langle 0, \frac{1}{2}\rangle\}$ (in particular, $\mathcal{A}$ is not quadro-classically hereditary). In this way, Theorem 6.32, Corollary 6.34 and Lemma 6.37 complete the argument. □

This provides an effective algebraic criterion of the structural completeness of C, whenever $\mathcal{A}$ is false-singular, the opposite case being analyzed in the next subparagraph. In view of the non-optional "false-singular" version of Example 6.21, the item (i) of Theorem 6.43 cannot be omitted. Likewise, its item (ii) cannot be omitted, in view of the optional version of Example 6.35, even if C is disjunctive. And what is more, its item (iii) cannot be omitted, even if C is both conjunctive and disjunctive, in view of Remark 2.8(i)(c), Corollary 6.34(ii) and the $\sim$-paraconsistent conjunctive disjunctive instances with classically hereditary characteristic matrices summarized in Paragraph 6.2.1.1. Furthermore, in view of Remark 2.8(i)(d) and Example 6.33, the item (iv) of Theorem 6.43 cannot be omitted, even if C is weakly disjunctive. Finally, its item (v) cannot equally be omitted, even if C is weakly disjunctive, in view of the optional version of:

Example 6.44. Let $\Sigma \triangleq \Sigma^{\intercal}_{\sim}$ with unary $\intercal$ and $\mathcal{A}$ both false-singular and [neither] involutive [(and so neither extra- nor quadro-classically hereditary) nor $\sim$-negative] (and so $\sim$-paraconsistent) with $\intercal^{\mathfrak{A}}(a) \triangleq \max(a, \frac{1}{2})$, for all $a \in A$, in which case $\intercal x_0$ is a theorem of C (and so this is weakly $(\intercal x_0)$-disjunctive) while $\mathcal{A}$ is not classically hereditary, for $\intercal^{\mathfrak{A}} 0 = \frac{1}{2} \notin 2 \ni 0$ [whereas $\Delta_2^- \cup \{\langle \frac{1}{2} + (i \cdot \frac{1}{2}), \frac{1}{2} + ((1-i) \cdot \frac{1}{2})\rangle \mid i \in 2\}$, being disjoint with $\Delta_A \cup \{0, \frac{1}{2}\}^2$, forms a subalgebra of $\mathfrak{A}^2$, and so neither $\mathcal{A}$ has a quasi-negation/"ternary equalizer" nor C has a proper $\sim$-paraconsistent extension, in view of Lemma 6.29(ii)⇒(i)]. And what is more, in the non-optional case, $\intercal\sim x_2$ is a ternary equalizer for $\mathcal{A}$, while, for no $j \in 2$, L_{3+j} forms a subalgebra of $\mathfrak{A}^2$, because $\intercal^{\mathfrak{A}^2} \langle 0, 1 - (j \cdot \frac{1}{2})\rangle = \langle \frac{1}{2}, 1 - (j \cdot \frac{1}{2})\rangle \notin L_{3+j} \ni \langle 0, 1 - (j \cdot \frac{1}{2})\rangle$, and so, by Theorem 6.43, C is structurally complete. On the other hand, in that case, $(A^2 \setminus \Delta_2) \supseteq \Delta_2^-$ forms a subalgebra of $\mathfrak{A}^2$, and so $\mathcal{A}$ has no binary semi-conjunction (in particular, is not weakly conjunctive, in view of Corollary 6.34). □

In this way, the characterization of the structural completeness given by Theorem 6.43 is minimal. In this connection, it is also remarkable that, though "the non-involutivity"/"existence of a binary semi-conjunction" of $\mathcal{A}$ subsumes the items (ii,iv/v) of Theorem 6.43, these cannot be collectively replaced by the single former stipulation, because there are structurally complete $\sim$-paraconsistent Σ-logics with subclassical negation $\sim$ and with involutive characteristic matrix having no binary semi-conjunction (and so not being weakly conjunctive), in view of the non-optional version of Example 6.44. In particular, structural completeness and weak conjunctivity do not imply one another, in view of $\sim$-paraconsistent conjunctive

instances with classically hereditary characteristic matrices summarized in Paragraph 6.2.1.1. Though Theorem 6.43 refutes the structural completeness of such instances, it equally shows that their uniform three-valued expansions (cf. Theorem 6.27) with a new nullary connective taking the value $\frac{1}{2}$ are structurally complete.
6.2.2.2.5. Structural completeness of weakly disjunctive paracomplete U3VLSN versus maximal paracompleteness and ternary anti-equalizers. Let $K_3 \triangleq (\Delta_2 \cup \{\langle\frac{1}{2},1\rangle\})$, $K_4 \triangleq (K_3 \cup \{\langle\frac{1}{2},0\rangle\})$ and $\mathcal{K}$ the submatrix of $\mathcal{A}^2$ generated by K_3, in which case $\mathcal{A}$ has no ternary anti-equalizer iff K is disjoint with Δ_2^-, while, providing $\mathcal{A}$ is classically hereditary, $K \subseteq (A \times 2)$, for $\pi_1[K_3] = 2$ forms a subalgebra of $\mathfrak{A}$, in which case K is disjoint with Δ_2^- iff either K_3 or K_4 forms a subalgebra of $\mathfrak{A}^2$, for $K_3 \subseteq K_4 = ((A\times 2)\setminus\Delta_2^-)$, whereas $(K_4 \setminus K_3) = \{\langle\frac{1}{2},0\rangle\}$ is a singleton, and so $\mathcal{A}$ has no ternary anti-equalizer iff either K_3 or K_4 forms a subalgebra of $\mathfrak{A}^2$.

Lemma 6.45. *Suppose C is $\sim$-subclassical but neither purely-inferential nor $\sim$-paraconsistent (in particular, $\mathcal{A}$ is truth-singular). Then, the following are equivalent:*

(i) *C is axiomatically equivalent to C^{PC};*
(ii) $(\mathrm{Fm}^1_\Sigma \cap (C^{\mathrm{PC}}(\varnothing) \setminus C(\varnothing))) = \varnothing$*;*
(iii) *The carrier of the subalgebra of $\mathfrak{A}^3$ generated by $\{\langle 0,1,\frac{1}{2}\rangle\}$ is disjoint with $\{\langle 1,1\rangle\} \times (A \setminus D^{\mathcal{A}})$;*
(iv) *The carrier of the subalgebra of $\mathfrak{A}^3$ generated by $\{\langle 0,1,\frac{1}{2}\rangle\}$ is disjoint with $(D^{\mathcal{A}})^2 \times (A \setminus D^{\mathcal{A}})$.*

Proof. In that case, $\sim^{\mathfrak{A}}[D^{\mathcal{A}}] = \{0\}$, for $\mathcal{A}$ is not $\sim$-paraconsistent, while, by Remark 2.4, there is some $\phi \in (\mathrm{Fm}^1_\Sigma \cap C(\varnothing))$, and so every $j \in 2$ is term-wise definable by $\sim^{j+1}\phi$ in $\mathfrak{A}$, whereas (ii/iii) is a particular case of (i/iv), respectively. Next, assume (ii) does not hold, in which case there is some $\varphi \in (\mathrm{Fm}^1_\Sigma \cap (C^{\mathrm{PC}}(\varnothing) \setminus C(\varnothing))) \neq \varnothing$ (in particular, C^{PC} is an inferentially consistent proper extension of C), and so, by Theorem 6.36, C is not $\sim$-classical. Then, by Theorems 6.25 and 6.32, $\mathcal{A}$ is classically hereditary, in which case φ is true in $\mathcal{A}_{\mathrm{PC}} = (\mathcal{A}\restriction 2)$, and so $\varphi^{\mathfrak{A}}(\frac{1}{2}) \notin D^{\mathcal{A}}$ (in particular, (iii) does not hold). Conversely, assume (iv) does not hold, in which case there is some $\xi \in \mathrm{Fm}^1_\Sigma$ such that $\xi^{\mathfrak{A}}[2] \subseteq D^{\mathcal{A}}$, while $\xi^{\mathfrak{A}}(\frac{1}{2}) \notin D^{\mathcal{A}}$, and so ξ is not true in $\mathcal{A}$ under $[x_0/\frac{1}{2}]$. Consider the following complementary cases:

- $\mathcal{A}$ is classically hereditary, in which case $\xi^{\mathfrak{A}}[2] \subseteq (D^{\mathcal{A}} \cap 2) = \{1\}$, and so, by Theorem 6.32(i), ξ is true in $(\mathcal{A}\restriction 2) = \mathcal{A}_{\mathrm{PC}}$ (in particular, $\xi \in (\mathrm{Fm}^1_\Sigma \cap (C^{\mathrm{PC}}(\varnothing) \setminus C(\varnothing))))$.
- $\mathcal{A}$ is not classically hereditary, in which case, since C is not $\sim$-paraconsistent, by Theorem 6.32(iii), $\theta^{\mathcal{A}} \in \mathrm{Con}(\mathfrak{A})$, and so, Theorem 6.32(ii), $h \triangleq \chi^{\mathcal{A}} \in \hom(\mathfrak{A}, \mathfrak{A}_{\mathrm{PC}})$. Then, as $h\restriction 2$ is diagonal, we have $\{1\} = h[D^{\mathcal{A}}] \supseteq h[\xi^{\mathfrak{A}}[2]] = \xi^{\mathfrak{A}_{\mathrm{PC}}}[h[2]] = \xi^{\mathfrak{A}_{\mathrm{PC}}}[2]$, in which case ξ is true in $\mathcal{A}_{\mathrm{PC}}$, and so $\xi \in (\mathrm{Fm}^1_\Sigma \cap (C^{\mathrm{PC}}(\varnothing) \setminus C(\varnothing)))$.

Thus, in any case (ii) does not hold. Finally, assume (i) does not hold, in which case there is some $\psi \in (C^{\mathrm{PC}}(\varnothing) \setminus C(\varnothing)) \neq \varnothing$, and so there is some $h \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{A})$ such that $h(\psi) \notin D^{\mathcal{A}}$. For each $a \in A$, set $N_a \triangleq \{i \in \omega \mid h(x_i) = a\}$. Let σ be the Σ-substitution extending $[x_l/\sim^{k+1}\phi; x_m/x_0]_{k\in 2, l\in N_k; m\in N_{\frac{1}{2}}}$ and $g \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{A})$ extend $[x_n/\frac{1}{2}]_{n\in\omega}$, in which case $h = (\sigma \circ g)$, and so, by the structurality of C^{PC}, $\sigma(\psi) \in (\mathrm{Fm}^1_\Sigma \cap (C^{\mathrm{PC}}(\varnothing) \setminus C(\varnothing)))$ (in particular, (ii) does not hold). □

Lemma 6.46. *Suppose C is $\sim$-subclassical, while $\mathcal{A}$ is truth-singular and has no ternary anti-equalizer. Let $\varphi \in (\mathrm{Fm}^1_\Sigma \cap (C^{\mathrm{PC}}(\varnothing) \setminus C(\varnothing)))$, $\mathcal{B}$ a truth-non-empty model of C not satisfying φ and C' the logic of $\mathcal{B}$. Then, $\mathcal{A} \in \mathrm{Mod}(C')$.*

Proof. In that case, C^{PC} is an inferentially consistent proper extension of C, and so, by Theorem 6.36, C is not $\sim$-classical. Hence, by Theorems 6.25 and 6.32, $\mathcal{A}$ is classically hereditary, in which case φ, being true in $\mathcal{A}_{\mathrm{PC}} = (\mathcal{A}{\restriction}2)$, is not true in $\mathcal{A}$ under $[x_0/\frac{1}{2}]$, and so $\varphi^{\mathfrak{A}}(\frac{1}{2}) = \frac{1}{2}$, for, otherwise, as $D^{\mathcal{A}} = \{1\}$, $\varphi(x_2)$ would be a ternary anti-equalizer for $\mathcal{A}$. And what is more, since φ is not true in $\mathcal{B}$, there is some $a \in B$ such that $\varphi^{\mathfrak{B}}(a) \notin D^{\mathcal{B}}$. Take any $b \in D^{\mathcal{B}} \neq \varnothing$. Then, by (2.14), the submatrix $\mathcal{D}$ of $\mathcal{B}$ generated by $\{a, b\}$ is a finitely-generated truth-non-empty model of C' (in particular, of its sublogic C), in which φ is not true under $[x_0/a]$. Therefore, by Lemma 3.7, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$, and some subdirect product $\mathcal{E} \in \mathbf{H}^{-1}(\mathcal{D}/\partial(\mathcal{D}))$ of it, in which case, by (2.14) and Remark 2.8(ii)(b), $\mathcal{E}$ is truth-non-empty model of C' not satisfying φ, and so there is some $c \in E$ such that $E \ni d \triangleq \varphi^{\mathfrak{E}}(c) \notin D^{\mathcal{E}}$. Then, $J \triangleq \{i \in I \mid \pi_i(c) = \frac{1}{2}\} \neq \varnothing$. Take any $e \in D^{\mathcal{E}} \neq \varnothing$, in which case, as $D^{\mathcal{A}} = \{1\}$, $E \ni e = (I \times \{1\})$, and so $E \ni f \triangleq {\sim}^{\mathfrak{E}} e = (I \times \{0\})$. Consider the following complementary cases:

- $J = I$, in which case $E \ni d = (I \times \{\frac{1}{2}\})$, and so, as $I = J \neq \varnothing$, $\{\langle g, I \times \{g\}\rangle \mid g \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{E} \in \mathrm{Mod}(C')$ (in particular, by (2.14), $\mathcal{A} \in \mathrm{Mod}(C')$).
- $J \neq I$, in which case, as $J \neq I$, $h : A^2 \to A^I$, $\langle j, k\rangle \mapsto ((J \times \{j\}) \cup ((I \setminus J) \times \{k\}))$ is injective. Then, $h(\langle 1|0, 1|0\rangle) = (e|f) \in E$ and $h(\langle \frac{1}{2}, 1\rangle) = d \in E$, in which case $h[K_3] \subseteq E$, and so $h{\restriction}K$ is an embedding of $\mathcal{K}$ into $\mathcal{E} \in \mathrm{Mod}(C')$. And what is more, since K is disjoint with Δ_2^-, for $\mathcal{A}$ has no ternary anti-equalizer, $(\pi_0{\restriction}K) \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{K}, \mathcal{A})$, for $\pi_0[K] = A$. Thus, by (2.14), $\mathcal{A} \in \mathrm{Mod}(C')$. ☐

Lemma 6.47. *Suppose C is non-$\sim$-paraconsistent (in particular, $\mathcal{A}$ is truth-singular), $\veebar$-disjunctive, $\sim$-subclassical and axiomatically-equivalent to C^{PC}. Then, $C = C^{\mathrm{PC}}$.*

Proof. In that case, by Lemma 3.11, C satisfies (3.2), and so (2.7) with $\sqsupset \triangleq \sqsupset^{\sim}_{\veebar}$. And what is more, by Lemma 6.18, $\mathcal{A}_{\mathrm{PC}} \in \mathrm{Mod}(C)$, being false-singular, is $\veebar$-disjunctive, in which case it, being $\sim$-negative, is $\sqsupset$-implicative, in view of Remark 2.8(i)(b), and so C^{PC}, being defined by the two-valued Σ-matrix $\mathcal{A}^{\mathrm{PC}}$, both is finitary and has DT with respect to $\sqsupset$. In this way, Lemma 4.10 and the fact that C is a sublogic of C^{PC} complete the argument. ☐

Theorem 6.48. *Suppose $\mathcal{A}$ is truth-singular (and so non-$\sim$-paraconsistent). Then, C is structurally complete iff the following hold:*

(i) *C has a theorem (i.e., the submatrix of $\mathcal{A}^3$ generated by $\{\langle 0, 1, \frac{1}{2}\rangle\}$ is truth-non-empty);*
(ii) *providing C is $\sim$-subclassical but not $\sim$-classical (in which case $\mathcal{A}$ is classically hereditary; cf. Theorems 6.25 and 6.32), the following hold:*
  (a) *C is not axiomatically equivalent to C^{PC} [in particular, disjunctive; cf. Lemma 6.47 {more specifically, implicative; cf. Theorem 3.5}] (i.e., the carrier of the subalgebra of $\mathfrak{A}^3$ generated by $\{\langle 0, 1, \frac{1}{2}\rangle\}$ is not disjoint with $\{\langle 1, 1\rangle\} \times \{0, \frac{1}{2}\}$; cf. Lemma 6.45(i)⇔(iii));*
  (b) *$\mathcal{A}$ has no ternary anti-equalizer (i.e., either K_3 or K_4 forms a subalgebra of $\mathfrak{A}^2$) ⟨in which case $\mathcal{A}$ is non-implicative, and so is C; cf. Corollary 6.19⟩.*

Proof. First, assume both of (i,ii) hold. Then, in case C is either $\sim$-classical or non-$\sim$-subclassical, by (i) and Theorem 6.36, C is structurally complete. Now, assume C is both non-$\sim$-classical and $\sim$-subclassical, in which case, by (ii)(a), it is not axiomatically-equivalent to C^{PC}, and so, by (i) and Lemma 6.45(ii)⇒(i), there is some $\varphi \in (\mathrm{Fm}^1_\Sigma \cap (C^{\mathrm{PC}}(\varnothing) \setminus C(\varnothing))) \neq \varnothing$, while, by (ii)(b), $\mathcal{A}$ has no ternary anti-equalizer. Consider any axiomatically-equivalent extension C' of C, in which case, by (i), $\varphi \notin T \triangleq C'(\varnothing) = C(\varnothing) \neq \varnothing$, and so, by the structurality of C', $\mathcal{B} \triangleq \langle \mathfrak{Fm}^{\omega}_{\Sigma}, T\rangle$ is a truth-non-empty model of C' (in particular, of its sublogic C), in which φ is not true under the diagonal Σ-substitution. Then, by Lemma 6.46, $\mathcal{A}$ is a model of C', for $\mathcal{B}$ is so, in which case $C' = C$, and so C is structurally complete. Conversely, assume either of (i,ii) does not hold. Consider the respective cases:

(i) does not hold, in which case, by Remark 2.9, C, being inferentially consistent, for $\mathcal{A}$ is both consistent and truth-non-empty, is not structurally complete.

(ii) does not hold, in which case C is both $\sim$-subclassical and non-$\sim$-classical (in particular, C^{PC} is a proper extension of C), and so is not structurally complete, whenever it is axiomatically-equivalent to C^{PC}. Otherwise, $\mathcal{A}$ has a ternary anti-equalizer, so, by Lemma 6.42, C is not structurally complete. □

This provides an effective purely-algebraic criterion of the structural completeness of C, whenever $\mathcal{A}$ is truth-singular. In view of the non-optional "truth-singular" version of Example 6.21, the item (i) of Theorem 6.48 cannot be omitted. Such equally concerns the subitem (b) of its item (ii) (in particular, the item (ii) itself), in view of existence of $\sim$-subclassical $\veebar$-disjunctive $(\veebar, \sim)$-paracomplete (and so non-$\sim$-classical with truth-singular characteristic matrices; cf. Remark 2.8(i)(d)) implicative (and so non-purely-inferential; cf. (2.5)) uniformly three-valued Σ-logics with subclassical negation $\sim$ like, e.g., Łukasiewicz' one (cf. Example 4.17 with $n = 3$) and IP^1 (cf. Subparagraph 6.2.1.1.4). Likewise, its subitem (a) cannot be omitted, while its optional stipulation of disjunctivity (as well as the regular one in Lemma 6.47) cannot be even weakened, in view of:

Example 6.49 (The disjunction-conjunction-implication-less fragment of G_3)**.** Let $\Sigma \triangleq \Sigma_{\sim,01}$ and $\mathcal{A}$ both truth-singular, non-involutive, non-$\sim$-negative with $(\mathfrak{A}{\restriction}\Sigma_{01}) \triangleq (\mathfrak{D}_{3,01}{\restriction}\Sigma_{01})$ (cf. Subparagraph 2.2.1.2.1), in which case, by Remark 2.8(ii)(a) and Theorems 6.25 and 6.32(i), C is not $\sim$-classical but is $\sim$-subclassical (in particular, C^{PC} is a proper extension of C) with $\mathcal{A}_{\mathrm{PC}} = (\mathcal{A}{\restriction}2)$, for $\mathcal{A}$ is classically hereditary, while $\top$ is a theorem of C, whereas K_3 forms a subalgebra of $\mathfrak{A}^2$, and so $\mathcal{A}$ has no ternary anti-equalizer as well as, by Remark 2.4, is weakly disjunctive. On the other hand, $K_5 \triangleq (2^2 \cup \{\langle \frac{1}{2}, 0\rangle\})$ forms a subalgebra of $\mathfrak{A}^2$, while $\pi_0{\restriction}K_{3|5}$ is a surjective homomorphism from $\mathcal{K}'_{3|5} \triangleq \langle \mathfrak{A}^2{\restriction}K_{3|5}, (\pi_1{\restriction}K_{3|5})^{-1}[\{1\}]\rangle$ onto $\mathcal{A}'_{\frac{1}{2}|0} \triangleq \langle \mathfrak{A}, \{1, \frac{1}{2}|0\}\rangle$, whereas $(\pi_1{\restriction}K_{3|5}) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{K}'_{3|5}, \mathcal{A}{\restriction}2)$, in which case by, (2.14,2.15), any $\varphi \in C^{\mathrm{PC}}(\varnothing)$ is true in both $\mathcal{A}'_{\frac{1}{2}}$ and $\mathcal{A}'_0$, and so in $\mathcal{A}$, for $D^{\mathcal{A}} = \{1\} = (D^{\mathcal{A}'_{\frac{1}{2}}} \cap D^{\mathcal{A}'_0})$. Then, C is axiomatically-equivalent to C^{PC}, and so is neither structurally complete nor disjunctive, in view of Lemma 6.47. □

In this way, the characterization of structural completeness given by Theorem 6.48 is minimal. Nevertheless, it can be enhanced for weakly $\veebar$-disjunctive $(\veebar, \sim)$-paracomplete uniformly three-valued Σ-logics with subclassical negation $\sim$, the rest of this subparagraph being devoted to this enhancement. We start from proving:

Theorem 6.50. *Suppose $\mathcal{A}$ is both weakly $\veebar$-disjunctive and $(\veebar, \sim)$-paracomplete as well as has [no] tautologies (i.e., is [not] extra-classically non-hereditary). Then, C is maximally [inferentially] $(\veebar, \sim)$-paracomplete iff, providing $\mathcal{A}$ is classically hereditary (i.e., C is $\sim$-subclassical), it has no ternary anti-equalizer (i.e., either K_3 or K_4 forms a subalgebra of $\mathfrak{A}^2$), in which case, providing C is {not} non-$\sim$-subclassical (i.e., $\mathcal{A}$ is {not} classically non-hereditary), it has no proper [inferentially] consistent extension {other than C^{PC} [and C^{PC}_{+0}], as well as is not implicative}.*

Proof. In that case, for each $j \in 2$, as $\{j, 1-j\} = 2 \ni 1 \in D^{\mathcal{A}}$, $(j \veebar^{\mathfrak{A}} \sim^{\mathfrak{A}} j) \in D^{\mathcal{A}}$, and so $(\frac{1}{2} \veebar^{\mathfrak{A}} \sim^{\mathfrak{A}} \frac{1}{2}) \notin D^{\mathcal{A}}$. Hence, $\{\frac{1}{2}, \sim^{\mathfrak{A}} \frac{1}{2}\}$ is disjoint with $D^{\mathcal{A}} \not\ni 0$, in which case $D^{\mathcal{A}} = \{1\}$, that is, $\mathcal{A}$ is truth-singular, and so is not $\sim$-paraconsistent.

Furthermore, by Remark 2.8(i)(d), C is not $\sim$-classical, and so, by Theorems 6.25 and 6.32, C, being non-$\sim$-paraconsistent, is $\sim$-subclassical iff $\mathcal{A}$ is classically hereditary, in which case $\mathcal{A}_{\mathrm{PC}} = (\mathcal{A}{\restriction}2)$.

Likewise, as $\mathcal{A}$ is truth-singular, $(\mathcal{A}{\restriction}\Sigma_{\sim}){\restriction}\{\frac{1}{2}\}$ is the only truth-empty submatrix of $\mathcal{A}{\restriction}\Sigma_{\sim}$, and so, by Corollary 3.10(i)⇔(iv), C has theorems iff $\mathcal{A}$ is not extra-classically hereditary.

Next, the "only if" part is by Lemma 6.42, for $\mathcal{A} \times (\mathcal{A}{\restriction}2)$ is truth-non-empty, while $\mathcal{A}\langle{\restriction}2\rangle$ is $\langle$not$\rangle$ $(\veebar, \sim)$-paracomplete, whenever $\mathcal{A}$ is classically hereditary.

Conversely, assume $\mathcal{A}$ has no ternary anti-equalizer, whenever it is classically hereditary. Consider any inferentially consistent extension C' of C and the following complementary cases:

- C is $\sim$-subclassical, in which case $\mathcal{A}$ is classically hereditary, and so has no ternary anti-equalizer. Consider the following complementary subcases:
  - C' is inferentially $(\veebar, \sim)$-paracomplete, in which case $(x_0 \veebar \sim x_0) \notin T \triangleq C'(x_1) \ni x_1$, and so, by the structurality of C', $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, T\rangle$ is a truth-non-empty $(\veebar, \sim)$-paracomplete model of C' (in particular, of its sublogic C). Then, by Lemma 6.46 with $\varphi = (x_0 \veebar \sim x_0)$, $C' = C$.
  - C' is not inferentially $(\veebar, \sim)$-paracomplete, in which case (2.11) is satisfied in it. Consider the following complementary subsubcases:
    - C' has a theorem ϕ, in which case, by the structurality of C' and (2.11)$[x_1/\phi]$, $x_0 \veebar \sim x_0$ is satisfied in C', and so, by Lemma 6.28, C' is an inferentially consistent extension of C^{PC}. Then, by Theorem 6.36, $C' = C^{\mathrm{PC}}$.
    - C' has no theorem, and so does its sublogic C. Let C'' be the closure operator over $\mathrm{Fm}^{\omega}_{\Sigma}$ dual to the closure system over $\mathrm{Fm}^{\omega}_{\Sigma}$ with basis $\mathcal{B} \triangleq ((\mathrm{img}\, C') \setminus \{\varnothing\})$, in which case $(C''{\restriction}\wp_{\infty\setminus 1}(\mathrm{Fm}^{\omega}_{\Sigma})) = (C'{\restriction}\wp_{\infty\setminus 1}(\mathrm{Fm}^{\omega}_{\Sigma}))$, and so, as the set Var_{ω} is infinite, while the set of all variables occurring in any Σ-formula is finite, C'' is structural, for C' is so. Then, C'' is an extension of C' (in particular, of its sublogic C) such that $C''(\varnothing) = (\mathrm{Fm}^{\omega}_{\Sigma} \cap \bigcap \mathcal{B})$, in which case $x_0 \veebar \sim x_0$ is satisfied in C'', and so, by Lemma 6.28, C'' is an inferentially consistent (for its $(\infty \setminus 1)$-extension C' is so) extension of C^{PC}. Hence, by Theorem 6.36, $C'' = C^{\mathrm{PC}}$, and so $C' = C''_{+0} = C^{\mathrm{PC}}_{+0}$.
- C is not $\sim$-subclassical, in which case, by Theorem 6.36, $C' = C$, for C is not $\sim$-paraconsistent.

Finally, Corollary 6.19 and the fact that any inferentially $(\veebar, \sim)$-paracomplete Σ-logic is inferentially consistent, while any Σ-logic with theorems is consistent/$(\veebar, \sim)$-paracomplete iff it is inferentially so, complete the argument. □

Since any Σ-logic axiomatically-equivalent to a $(\veebar, \sim)$-paracomplete one is $(\veebar, \sim)$-paracomplete, by Remark 2.8(i)(d), Theorems 6.25, 6.32, 6.48, 6.50 (as well as the first three paragraphs of its proof) and Corollary 3.10(i)⇔(iv), we eventually get:

Corollary 6.51. *Suppose C is both weakly $\veebar$-disjunctive and $(\veebar, \sim)$-paracomplete. Then, the following are equivalent:*

(i) *C is structurally complete;*
(ii) *C is maximally $(\veebar, \sim)$-paracomplete;*
(iii) *the following hold:*
  (a) *$\mathcal{A}$ is not extra-classically hereditary (i.e., C has a theorem);*

(b) *providing $\mathcal{A}$ is classically hereditary (i.e., C is $\sim$-subclassical), it has no ternary anti-equalizer (i.e., either K_3 or K_4 forms a subalgebra of $\mathfrak{A}^2$),*

in which case, providing C is {not} non-$\sim$-subclassical (i.e., $\mathcal{A}$ is {not} classically non-hereditary), it has no proper consistent extension {other than C^{PC}, as well as is not implicative}.

Theorem/Corollary 6.50/6.51 provides a purely-algebraic criterion of "maximal [inferential] $(\veebar,\sim)$-paracompleteness"/"structural completeness" of weakly $\veebar$-disjunctive $(\veebar,\sim)$-paracomplete uniform three-valued Σ-logics with subclassical negation $\sim$ "and with[out] theorems"/, covering positively both $G_3^{\langle *\rangle}$ {with $K_{3(+1)}$ (not) forming a subalgebra of $\mathfrak{A}^2$} and $KL_{3,01}$ [as well as /negatively KL_3 with extra-classically hereditary characterisic matrix] {with $K_{4(-1)}$ (not) forming a subalgebra of $\mathfrak{A}^2$}, and so demonstrating the necessity of regarding both K_3 and K_4 as well as yielding a new insight into the non-implicativity {regardless to any connective} of the weakly $\supset$-implicative G_3. (In view of Lemma 4.24 of [23] with $B = DM_{3,1}$ and $\Sigma = \Sigma_{\sim,+,01}$, $KL_{3,01}$ is axiomatically-equivalent to $B_{4,01}$, in which case the former, being structurally complete, is the structural completion of the latter.) Likewise, it negatively covers implicative (and so non-purely-inferential; cf. (2.5)) $\veebar$-disjunctive $(\veebar,\sim)$-paracomplete uniform three-valued Σ-logics with subclassical negation $\sim$ like Łukasiewicz' one (cf. Example 4.17 with $n = 3$) and IP^1 (cf. Subparagraph 6.2.1.1.4).

6.2.2.3. Self-extensionality versus discriminating endomorphisms. A *(truth-)discriminating operator/endomorphism on/of* $\mathcal{A}$ is any $h \in (A^A/\hom(\mathfrak{A},\mathfrak{A}))$ such that $\chi^{\mathcal{A}}(h(\frac{1}{2})) \neq \chi^{\mathcal{A}}(h(\Bbbk^{\mathcal{A}}))$, in which case $h(\frac{1}{2}) \neq h(\Bbbk^{\mathcal{A}})$, and so h is neither diagonal nor singular, the set of all them being denoted by $(\partial/\eth)(\mathcal{A})$, respectively. Then, since $\mathrm{img}[\theta^{\mathcal{A}} \setminus \Delta_A] = \{\{\frac{1}{2},\Bbbk^{\mathcal{A}}\}\}$, by Example 4.2, Corollary 4.12 and Theorem 6.25(iii)⇒(i), we have:

Corollary 6.52. *[Providing $\mathcal{A}$ is either implicative or both conjunctive and disjunctive] C is self-extensional if[f] either it is $\sim$-classical or $\eth(\mathcal{A}) \neq \varnothing$.*

Though there are $3^3 = 27$ unary operations on A, only few of them may be discriminating operators/endomorphisms on/of $\mathcal{A}$. More precisely, let $h_{+|-,a} \triangleq (\Delta_2^{+|-} \cup \{\langle\frac{1}{2},a\rangle\}) \in A^A$, where $a \in A$, $\mathcal{H} \triangleq (\bigcup_{a\in A}\{h_{+,a},h_{-,a}\})$ and $\mathcal{H}^{\mathcal{A}} \triangleq (\{h_{-,a} \mid a \in A, \chi^{\mathcal{A}}(a) = \Bbbk^{\mathcal{A}}\} \cup \{h_{+,1-\Bbbk^{\mathcal{A}}}\})$. Clearly,

$$(\mathcal{H} \cap \partial(\mathcal{A})) = \mathcal{H}^{\mathcal{A}}. \tag{6.5}$$

Conversely, since $\eth(\mathcal{A}) = (\partial(\mathcal{A}) \cap \hom(\mathfrak{A},\mathfrak{A}))$, by (6.5) and Lemma 6.22(i) with $\mathcal{D} = \mathcal{A} = \mathcal{B}$, we have:

Corollary 6.53. $\eth(\mathcal{A}) \subseteq \mathcal{H}$. *In particular,* $\eth(\mathcal{A}) = (\mathcal{H}^{\mathcal{A}} \cap \hom(\mathfrak{A},\mathfrak{A}))$.

Combining Corollaries 6.52 and 6.53, we eventually get:

Theorem 6.54. *[Providing $\mathcal{A}$ is either implicative or both conjunctive and disjunctive] C is self-extensional if[f] either it is $\sim$-classical or $(\mathcal{H}^{\mathcal{A}} \cap \hom(\mathfrak{A},\mathfrak{A})) \neq \varnothing$.*

This yields a quite effective purely-algebraic criterion of the self-extensionality of C with either implicative or both conjunctive and disjunctive $\mathcal{A}$ that can inevitably be enhanced a bit more under separate studying the alternatives involved excluding *a priori* some elements of $\mathcal{H}^{\mathcal{A}}$ from $\eth(\mathcal{A})$ (i.e., from $\hom(\mathfrak{A},\mathfrak{A})$; cf. Corollary 6.53), because, under the stipulation of C's being both self-extensional and non-$\sim$-classical, the alternatives under considerations are disjoint, as it is shown below.

6.2.2.3.1. Self-extensionality versus equational truth-definitions.

Lemma 6.55. *Let $\mho$ be an equational truth definition for $\mathcal{A}$. Suppose $\mathcal{A}$ is either false-singular or $\sqsupset$-implicative, while C is not $\sim$-classical. Then, any non-singular endomorphism h of $\mathfrak{A}$ is diagonal. In particular, providing $\mathcal{A}$ is either implicative or both conjunctive and disjunctive, C is not self-extensional.*

Proof. Then, for any $a \in A$, we have $(a \in D^{\mathcal{A}}) \Leftrightarrow (\mathfrak{A} \models (\bigwedge \mho)[x_0/a]) \Rightarrow (\mathfrak{A} \models (\bigwedge \mho)[x_0/h(a)]) \Leftrightarrow (h(a) \in D^{\mathcal{A}})$, in which case $h \in \hom(\mathcal{A}, \mathcal{A})$ (in particular, $h(1) \neq 0$, for $1 \in D^{\mathcal{A}} \not\ni 0$), and so, by Lemma 6.22(i) with $\mathcal{D} = \mathcal{A} = \mathcal{B}$, $h \restriction 2$ is diagonal. Therefore, if $h(\frac{1}{2})$ was equal to $\Bbbk^{\mathcal{A}}$, then h would be equal to $\chi^{\mathcal{A}}$, in which case $\theta^{\mathcal{A}} = (\ker h)$ would be a congruence of $\mathfrak{A}$, and so, by Theorem 6.25, C would be $\sim$-classical. Hence, in case $\mathcal{A}$ is false-singular, $h(\frac{1}{2}) = \frac{1}{2}$, for $\frac{1}{2} \in D^{\mathcal{A}} \not\ni 0$. Otherwise, $\mathcal{A}$ is $\sqsupset$-implicative, in which case $(\frac{1}{2} \sqsupset^{\mathfrak{A}} 0) = 1$ and $(1 \sqsupset^{\mathfrak{A}} 0) \neq 1$, and so $h(\frac{1}{2}) = \frac{1}{2}$, for otherwise, we would have $h(\frac{1}{2}) = 1$, in which case we would get $1 \neq 1$. Thus, in any case, $h(\frac{1}{2}) = \frac{1}{2}$, and so h is diagonal. In this way, Corollary 4.13 and Theorem 6.25(iii)⇒(i) complete the argument. □

This "equational truth definition" analogue of Corollary 4.15 provides another and much more transparent insight into the non-self-extensionality of the instances discussed in Example 4.18 and summarized below. In this connection, we first have:

Corollary 6.56. *Suppose $\mathcal{A}$ is both $\sqsupset$-implicative and either weakly $\overline{\wedge}$-conjunctive (in particular, $\wr$-negative with $\overline{\wedge} = \uplus^{\wr}_{\sqsupset}$; cf. Remark 2.8*(i)(a)*) or truth-singular. Then, $\mathcal{A}$ has a finitary equational truth-definition. In particular, C is not self-extensional, unless it is $\sim$-classical.*

Proof. The case, when $\mathcal{A}$ is truth-singular, is due to Remark 4.14(iv). Otherwise, $\mathcal{A}$ is weakly $\overline{\wedge}$-conjunctive, while $\{\frac{1}{2}\}$ does [not] form a subalgebra of $\mathfrak{A}$ [that is, there is some $\varphi \in \mathrm{Fm}^1_\Sigma$ such that $\varphi^{\mathfrak{A}}(a) \in 2$], so $\{(x_0 \sqsupset \phi) \approx \phi\}$ with $\phi \triangleq (\psi[\overline{\wedge}(\psi[x_0/\varphi])])$ and $\psi \triangleq (x_0 \mathbin{\overline{\wedge}} {\sim} x_0)$ is a finitary equational truth definition for $\mathcal{A}$. In this way, Lemma 6.55 completes the argument. □

This is why the contexts of the next two subparagraphs are disjoint, whenever C is self-extensional but not $\sim$-classical. Before coming to discussing them, we provide practically immediate applications of the above results of this subparagraph to some of the logics specified in Paragraph 6.2.1.1.

Remark 6.57. Suppose $\mathcal{A}$ is both $\sim$-paraconsistent (and so false-singular), conjunctive and $\veebar$-disjunctive as well as both classically- and extra-classically-hereditary. Then, $\{x_0 \approx (x_0 \veebar {\sim} x_0)\}$ is an equational truth definition for $\mathcal{A}$, so, by Remark 2.8(i)(c) and Lemma 6.55, C is not self-extensional. □

This subsumes disjunctive conjunctive $\sim$-paraconsistent LP and HZ, providing a more transparent insight into the non-self-extensionality of them than that given by Example 4.18. Likewise, $[I]P^1$ is subsumed by:

Remark 6.58. Suppose $\mathcal{A}$ is both classically-valued and $\diamond$-conjunctive/-disjunctive /(in particular, $\sqsupset$-implicative with $\diamond = \uplus_{\sqsupset}$). Then, it is $\wr$-negative, where $\wr x_0 \triangleq {\sim}(x_0 \diamond x_0)$, in which case, by Remark 2.8(i)(a), $\mathcal{A}$ is both $\overline{\wedge}$-conjunctive and $\veebar$-disjunctive, where $\overline{\wedge} \triangleq \diamond^{/\wr}$ and $\veebar \triangleq \diamond^{\wr/}$, and so, by Remark 2.8(i)(b), $\mathcal{A}$ is $\sqsupset^{\wr}_{\veebar}$-implicative. On the other hand, as $\frac{1}{2} \notin 2$, any idempotent binary operation on A, being term-wise definable in $\mathfrak{A}$, is so by either x_0 or x_1, in which case it is not symmetric, for A is not a singleton, and so $\mathfrak{A}$ is not a semi-lattice (in particular, is not a [distributive] lattice). And what is more, $\{((x_0 \sqsupset^{\wr}_{\veebar} x_0) \sqsupset^{\wr}_{\veebar} x_0) \approx (x_0 \sqsupset^{\wr}_{\veebar} x_0)\}$ is a finitary equational truth definition for $\mathcal{A}$, so, providing $\mathcal{A}$ is not $\sim$-negative (in which case it is $\sim$-paraconsistent|$(\underline{\vee},\sim)$-paracomplete, whenever it is false-|truth-singular), so, by Remark 2.8(i)(c|d) and Lemma 6.55, C is not self-extensional. $\square$

6.2.2.3.2. Conjunctive U3VLSN.

Lemma 6.59. *Let $\mathcal{B}$ be a consistent/truth-non-empty weakly $\diamond$-conjunctive/-disjunctive Σ-matrix. Suppose $\mathfrak{B}$ is a $\diamond$-semi-lattice with bound. Then, $\beta^{\mathfrak{B}}_{\diamond} \notin / \in D^{\mathcal{B}}$.*

Proof. By the weak $\diamond$-conjunctivity/-disjunctivity of $\mathcal{B}$, we do have $\beta^{\mathfrak{B}}_{\diamond} = (\beta^{\mathfrak{B}}_{\diamond} \diamond^{\mathfrak{B}} a) \notin / \in D^{\mathcal{B}}$, where $a \in ((B \setminus D^{\mathcal{B}})/D^{\mathcal{B}}) \neq \varnothing$. $\square$

Lemma 6.60. *Suppose C is weakly $\bar{\wedge}$-conjunctive. Then, $\mathfrak{A}$ is a $\bar{\wedge}$-semi-lattice with bound such that the following hold:*

(i) $(0 \,\bar{\wedge}^{\mathfrak{A}}\, 1) = \beta^{\mathfrak{A}}_{\bar{\wedge}}$;

(ii) $\frac{1}{2} \leq^{\mathfrak{A}}_{\bar{\wedge}} 1$;

(iii) [providing $\eth(\mathcal{A}) \neq \varnothing$, (g)$\Rightarrow$](a)$\Rightarrow$(b)$\Rightarrow$(c)$\Leftrightarrow$(d)$\Leftrightarrow$(e)$\Leftrightarrow$(f)$\Rightarrow$(g)$\Rightarrow$(h)[$\Rightarrow$(f)], where:

  (a) $h_{+,1-\Bbbk^{\mathcal{A}}} \in \hom(\mathfrak{A},\mathfrak{A})$;

  (b) $\mathcal{A}$ is classically-hereditary;

  (c) $\beta^{\mathfrak{A}}_{\bar{\wedge}} = 0$;

  (d) $0 \leq^{\mathfrak{A}}_{\bar{\wedge}} \frac{1}{2}$;

  (e) $0 \leq^{\mathfrak{A}}_{\bar{\wedge}} 1$;

  (f) $\mathcal{A}$ is not involutive;

  (g) $h_{-,a} \in \hom(\mathfrak{A},\mathfrak{A})$, for no $a \in A$;

  (h) $h_{-,\frac{1}{2}} \notin \hom(\mathfrak{A},\mathfrak{A})$;

(iv) $\mathcal{A}$ is not $\sim$-negative, unless $\eth(\mathcal{A}) = \varnothing$.

Proof. In that case, by Theorem 4.6(i)$\Rightarrow$(iv), $\mathfrak{A}$, being finite, is a $\bar{\wedge}$-semi-lattice with bound, so, by Lemma 6.59, $\beta^{\mathfrak{A}}_{\bar{\wedge}} \notin D^{\mathcal{A}}$. Let $\xi_{0[+1]} \triangleq [\sim]x_0$ as well as both $\phi_k \triangleq \xi_k(x_0 \,\bar{\wedge} \sim x_0)$ and $\psi_k \triangleq \phi_k(\sim x_0)$, where $k \in 2$.

(i) In case $\beta^{\mathfrak{A}}_{\bar{\wedge}} = 0$, we have $0 = \beta^{\mathfrak{A}}_{\bar{\wedge}} \leq^{\mathfrak{A}} 1$, and so get $(0 \,\bar{\wedge}^{\mathfrak{A}}\, 1) = 0 = \beta^{\mathfrak{A}}_{\bar{\wedge}}$. Otherwise, as $1 \in D^{\mathcal{A}}$, we have $D^{\mathcal{A}} \not\ni \beta^{\mathfrak{A}}_{\bar{\wedge}} = \frac{1}{2}$, in which case $\mathcal{A}$ is truth-singular, and so is non-$\sim$-paraconsistent, that is, C is so. Then, by (2.10) and the conjunctivity of C, we have $x_1 \in C(\phi_0)$, in which case, by Theorem 4.6(i)$\Rightarrow$(iv), we get $\beta^{\mathfrak{A}}_{\bar{\wedge}} \leq^{\mathfrak{A}} (0 \,\bar{\wedge}^{\mathfrak{A}}\, 1) = \phi^{\mathfrak{A}}_0(0) \leq^{\mathfrak{A}}_{\bar{\wedge}} \beta^{\mathfrak{A}}_{\bar{\wedge}}$, and so eventually get $(0 \,\bar{\wedge}^{\mathfrak{A}}\, 1) = \beta^{\mathfrak{A}}_{\bar{\wedge}}$.

(ii) Consider the following complementary cases:

- $\mathcal{A}$ is is false-singular, in which case, by (i), for each $k \in 2$, $\phi^{\mathfrak{A}}_0(k) = \phi^{\mathfrak{A}}_0(0) = \beta^{\mathfrak{A}}_{\bar{\wedge}} = 0$, and so $(\phi|\psi)^{\mathfrak{A}}_1(k) = 1 \in D^{\mathcal{A}}$. Consider the following complementary subcases:
  - $\sim^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2}$, in which case $\phi^{\mathfrak{A}}_1(\frac{1}{2}) = \frac{1}{2} \in D^{\mathcal{A}}$, for $\mathcal{A}$ is false-singular, and so ϕ_1 is true in $\mathcal{A}$ (in particular, $\phi_1 \in C(x_1)$). Then, by Theorem 4.6(i)$\Rightarrow$(iv), $\frac{1}{2} \leq^{\mathfrak{A}}_{\bar{\wedge}} \phi^{\mathfrak{A}}_1(0) = 1$.
  - $\sim^{\mathfrak{A}}\frac{1}{2} \neq \frac{1}{2}$, that is, $\sim^{\mathfrak{A}}\frac{1}{2} \in 2$, in which case $\psi^{\mathfrak{A}}_1(\frac{1}{2}) = \phi^{\mathfrak{A}}_1(\sim^{\mathfrak{A}}\frac{1}{2}) = 1 \in D^{\mathcal{A}}$, and so ψ_1 is true in $\mathcal{A}$ (in particular, $\psi_1 \in C(x_1)$). Then, by Theorem 4.6(i)$\Rightarrow$(iv), $\frac{1}{2} \leq^{\mathfrak{A}}_{\bar{\wedge}} \psi^{\mathfrak{A}}_1(0) = 1$.
- $\mathcal{A}$ is truth-singular, in which case it is non-$\sim$-paraconsistent, that is, C is so, and so, by (2.10) and the $\bar{\wedge}$-conjunctivity of C, $x_1 \in C(\phi_0)$. Consider the following complementary subcases:
  - $\frac{1}{2}$ is equal to either $\beta^{\mathfrak{A}}_{\bar{\wedge}}$ or $\sim^{\mathfrak{A}}\frac{1}{2}$, in which case we have $\frac{1}{2} = \phi^{\mathfrak{A}}_0(\frac{1}{2})$, and so, by Theorem 4.6(i)$\Rightarrow$(iv), get $\frac{1}{2} \leq^{\mathfrak{A}}_{\bar{\wedge}} 1$, for $x_1 \in C(\phi_0)$.
  - $\beta^{\mathfrak{A}}_{\bar{\wedge}} \neq \frac{1}{2} \neq \sim^{\mathfrak{A}}\frac{1}{2}$, in which case, as $1 \in D^{\mathcal{A}}$, by (i), for each $k \in 2$, $\phi^{\mathfrak{A}}_0(k) = (0 \,\bar{\wedge}^{\mathfrak{A}}\, 1) = \beta^{\mathfrak{A}}_{\bar{\wedge}} = 0$, and so $(\phi|\psi)^{\mathfrak{A}}_1(k) = 1 \in D^{\mathcal{A}}$ (in

particular, $\psi_1^{\mathfrak{A}}(\frac{1}{2}) = \phi_1^{\mathfrak{A}}(\sim^{\mathfrak{A}}\frac{1}{2}) = 1 \in D^{\mathcal{A}}$). Then, ψ_1 is true in $\mathcal{A}$, in which case $\psi_1 \in C(x_1)$, and so, by Theorem 4.6(i)⇒(iv), $\frac{1}{2} \leq^{\mathfrak{A}}_{\bar{\wedge}} \psi_1^{\mathfrak{A}}(0) = 1$.

(iii) First, (d/h) is a particular case of (c/g), while (d/e)⇒(e/c) is by (ii/i), whereas (b)⇒(e) is by the $\bar{\wedge}$-conjunctivity of $\mathcal{A}$ and the fact that $1 \in D^{\mathcal{A}} \not\ni 0$. Next, (a)⇒(b) is by the fact that $\mathrm{img}(h_{+,1-\Bbbk^{\mathcal{A}}}) = 2$. Further, assume (f) holds, in which case $l \triangleq \sim^{\mathfrak{A}}\frac{1}{2} \in 2$, and so $\xi^{\mathfrak{A}}_{1-l}(\frac{1}{2}) = 1 \in D^{\mathcal{A}}$. We prove (e) by contradiction. For suppose (e) does not hold, in which case $\beta^{\mathfrak{A}}_{\bar{\wedge}} \neq 0$, and so, by Lemma 6.59, $\beta^{\mathfrak{A}}_{\bar{\wedge}} = \frac{1}{2}$, for $1 \in D^{\mathcal{A}}$ (in particular, $\phi_0^{\mathfrak{A}}(\frac{1}{2}) = \frac{1}{2}$). Likewise, by (i), for each $k \in 2$, $\phi_0^{\mathfrak{A}}(k) = (0 \mathbin{\bar{\wedge}^{\mathfrak{A}}} 1) = \beta^{\mathfrak{A}}_{\bar{\wedge}} = \frac{1}{2}$, in which case ϕ_{1-l} is true in $\mathcal{A}$, and so $\phi_{1-l} \in C(x_1)$. Then, by Theorem 4.6(i)⇒(iv), $0 \leq^{\mathfrak{A}}_{\bar{\wedge}} \phi^{\mathfrak{A}}_{1-l}(0) = 1$. Thus, (e) holds. [Conversely, assume (f) does not hold, in which case $\sim^{\mathfrak{A}}a = (1-a)$, for all $a \in A$. Take any $h \in \eth(\mathfrak{A}) \neq \varnothing$, in which case it is neither diagonal nor singular, and so, by Lemma 6.22, $(h{\restriction}2) \in \{\Delta_2^+, \Delta_2^-\}$. Then, we have $h(\frac{1}{2}) = h(\sim^{\mathfrak{A}}\frac{1}{2}) = \sim^{\mathfrak{A}}h(\frac{1}{2}) = (1 - h(\frac{1}{2}))$, in which case we get $h(\frac{1}{2}) = \frac{1}{2}$, and so $h = h_{-,\frac{1}{2}}$, for, otherwise, h would be diagonal. Thus, (h)⇒(f) holds.] Now, assume (e) holds (that is, (c) does so), in which case, for each $k \in 2$, $\phi_0^{\mathfrak{A}}(k) = (0 \mathbin{\bar{\wedge}^{\mathfrak{A}}} 1) = 0$, and so $\phi_1^{\mathfrak{A}}(k) = 1 \in D^{\mathcal{A}}$. We prove (f) by contradiction. For suppose $\sim^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2}$, in which case $\phi_0^{\mathfrak{A}}(\frac{1}{2}) = \frac{1}{2}$, and so $\phi_1^{\mathfrak{A}}(\frac{1}{2}) = \frac{1}{2}$. Consider the following complementary cases:

- $\mathcal{A}$ is false-singular, in which case $\phi_1^{\mathfrak{A}}(\frac{1}{2}) = \frac{1}{2} \in D^{\mathcal{A}}$, and so ϕ_1 is true in $\mathcal{A}$ (in particular, $\phi_1 \in C(x_1)$). Then, by Theorem 4.6(i)⇒(iv), $1 \leq^{\mathfrak{A}}_{\bar{\wedge}} \phi_1^{\mathfrak{A}}(\frac{1}{2}) = \frac{1}{2}$, in which case, by (ii), $\frac{1}{2} = 1$, and so $\frac{1}{2} \in 2$.
- $\mathcal{A}$ is truth-singular, in which case it is not $\sim$-paraconsistent, and so, by (2.10) and the $\bar{\wedge}$-conjunctivity of C, $x_1 \in C(\phi_0)$. Then, by Theorem 4.6(i)⇒(iv), $\frac{1}{2} = \phi_0^{\mathfrak{A}}(\frac{1}{2}) \leq^{\mathfrak{A}}_{\bar{\wedge}} 0$, in which case, by (c), $\frac{1}{2} = 0$, and so $\frac{1}{2} \in 2$.

Thus, as $\frac{1}{2} \notin 2$, (f) does hold. Furthermore, if any $h : A \to A$ with $(h{\restriction}2) = \Delta_2^-$ was an endomorphism of $\mathfrak{A}$, then, by (e), we would have $1 = h(0) = h(0 \mathbin{\bar{\wedge}^{\mathfrak{A}}} 1) = (h(0) \mathbin{\bar{\wedge}^{\mathfrak{A}}} h(1)) = (1 \mathbin{\bar{\wedge}^{\mathfrak{A}}} 0) = (0 \mathbin{\bar{\wedge}^{\mathfrak{A}}} 1) = 0$, and so (g) holds. [Finally, (g)⇒(a) is by (6.5) and Lemma 6.22, for $\eth(\mathcal{A}) = (\partial(\mathcal{A}) \cap \hom(\mathfrak{A}, \mathfrak{A}))$.]

(iv) Assume $\eth(\mathcal{A}) \neq \varnothing$. Then, $\mathcal{A}$ is not $\sim$-negative, whenever it is involutive. Otherwise, by (iii)(f)⇒(a), $h \triangleq h_{+,1-\Bbbk^{\mathcal{A}}} \in \hom(\mathfrak{A}, \mathfrak{A})$, in which case, if $\mathcal{A}$ was $\sim$-negative, then we would have $\sim^{\mathfrak{A}}\frac{1}{2} = (1 - \Bbbk^{\mathcal{A}})$, and so would get $2 \ni \Bbbk^{\mathcal{A}} = \sim^{\mathfrak{A}}(1 - \Bbbk^{\mathcal{A}}) = \sim^{\mathfrak{A}}h(\frac{1}{2}) = h(\sim^{\mathfrak{A}}\frac{1}{2}) = h(1 - \Bbbk^{\mathcal{A}}) = (1 - \Bbbk^{\mathcal{A}})$. □

Theorem 6.61. *Suppose C is $\bar{\wedge}$-conjunctive, non-$\sim$-classical and self-extensional. Then, $\eth(\mathcal{A}) \neq \varnothing$.*

Proof. Then, by Theorem 6.25, $\mathcal{A}$ is hereditarily simple, while, by Theorem 4.6(i)⇒(iv) and Lemma 6.59, $\mathfrak{A}$, being finite, is a $\bar{\wedge}$-semi-lattice with bound $\beta^{\mathfrak{A}}_{\bar{\wedge}} \notin D^{\mathcal{A}}$, in which case, as $\frac{1}{2} \notin 2 \ni \Bbbk^{\mathcal{A}}$ (in particular, $\frac{1}{2} \neq \Bbbk^{\mathcal{A}}$), by the commutativity identity for $\bar{\wedge}$, there are some $\bar{a} \in (\{\frac{1}{2}, \Bbbk^{\mathcal{A}}\}^2 \setminus \Delta_A)$ and some $i \in 2$ such that $a_{1-i} \neq (a_i \mathbin{\bar{\wedge}^{\mathfrak{A}}} a_{1-i})$, and so $\mathcal{B} \triangleq \langle \mathfrak{A}, F\rangle$, where $a_i \in F \triangleq \{b' \in A \mid a_i \leq^{\mathfrak{A}}_{\bar{\wedge}} b'\} \not\ni a_{1-i}$, being both truth-non-empty and $\bar{\wedge}$-conjunctive, is a finite consistent truth-non-empty model of C. Then, as 2 forms a subalgebra of $\mathfrak{A}{\restriction}\Sigma_\sim$, by Remark 2.8(ii)(b), Lemmas 3.7, 6.24(i,ii) with $\Sigma' = \Sigma_\sim$ and the conjunctivity of $\mathcal{A}$, $((\mathcal{A}{\restriction}\Sigma_\sim){\restriction}2)$, being $\sim$-classical, belongs to $\mathbf{I}(\mathbf{S}(\mathbf{H}^{-1}(\mathbf{H}(\mathcal{B}{\restriction}\Sigma_\sim))))$, in which case, by (2.14), $\sim$ is a subclassical negation for the logic C' of $\mathcal{B}$, and so, by Theorem 6.15, $\mathcal{B}$, being three-valued, is $\sim$-super-classical. Let $\mathcal{D}$ be the canonization of $\mathcal{B}$, in which case they are isomorphic, and so, by (2.14), C' is defined by $\mathcal{D}$. Consider the following complementary cases:

- C' is $\sim$-classical, in which case, as it is $\overline{\wedge}$-conjunctive, for its sublogic C is so, by Theorem 6.25, $\mathcal{D}$ is a strictly surjectively homomorphic counter-image of a $\sim$-classical Σ-matrix $\mathcal{E}$, and so is $\mathcal{B}$, being isomorphic to $\mathcal{D}$. Then, by (2.14), $\mathcal{E}$ is a $\sim$-classical model of C, for $\mathcal{B} \in \mathrm{Mod}(C)$, in which case, by Theorem 6.32, $\mathcal{A}$ is classically hereditary, $\mathcal{A}{\restriction}2$ being isomorphic to $\mathcal{E}$, and so $\mathcal{B}$ is a strictly [surjectively] homomorphic counter-image of $\mathcal{A}[{\restriction}2]$.
- C' is not $\sim$-classical, in which case, by Theorem 6.25, $\mathcal{D}$, being canonically $\sim$-super-classical and defining C', is simple, and so is $\mathcal{B}$, being isomorphic to $\mathcal{D}$, in view of Remark 2.6(iii). Hence, by Lemma 3.7, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$, some subdirect product $\mathcal{G}$ of it and some $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{G},\mathcal{B})$, in which case, by Remark 2.8(ii)(b), $\mathcal{G}$ is both consistent and truth-non-empty, for $\mathcal{B}$ is so, and so, by Lemma 6.24(i), $a \triangleq (I\times\{1\}) \in G \ni b \triangleq (I\times\{0\})$. We prove, by contradiction, that $\mathcal{A}$ is truth-singular. For suppose it is false-singular, in which case, by Lemma 6.59, $0 = \beta^{\mathfrak{A}}_{\overline{\wedge}} \leq^{\mathfrak{A}}_{\overline{\wedge}} 1$, and so, by Lemma 6.60(ii)/(iii)(c)$\Rightarrow$(f), $(1 = \delta\beta^{\mathfrak{A}}_{\overline{\wedge}})/(\sim^{\mathfrak{A}}\frac{1}{2} \in 2)$, respectively. Then, $a_i \neq \frac{1}{2}$, for, otherwise, we would have $\frac{1}{2} = a_i \not\leq^{\mathfrak{A}}_{\overline{\wedge}} a_{1-i} = \Bbbk^{\mathcal{A}} = 1 = \delta\beta^{\mathfrak{A}}_{\overline{\wedge}}$. Hence, $a_i = \Bbbk^{\mathcal{A}} = 1$, in which case $D^{\mathcal{B}} = \{1\}$, for $1 = \delta\beta^{\mathfrak{A}}_{\overline{\wedge}}$, and so $\mathcal{B}$ is a finite, truth-singular, consistent, truth-non-empty model of C, in which (2.10) is not true under $[x_0/1, x_1/0]$. Therefore, by Remark 2.8(ii)(c) and Lemma 3.7, $\mathcal{A}$, being finite and simple but not truth-singular, is not a model of C', for truth-singularity is clearly preserved under $\mathbf{P}$, in which case, by Theorem 6.32(ii), $\mathcal{A}$, being conjunctive, is classically hereditary. Then, as $a \in D^{\mathcal{G}}$, for $1 \in D^{\mathcal{A}}$, $g(a) = 1$, in which case $g(b) = g(\sim^{\mathfrak{G}} a) = \sim^{\mathfrak{A}} g(a) = 0$, and so $g[\{a,b\}] = 2$. Furthermore, there is some $c \in G$ such that $g(c) = \frac{1}{2} \notin D^{\mathcal{B}}$, in which case $c \notin D^{\mathcal{G}}$, and so there is some $j \in I$ such that $\pi_j(c) = 0$, for $\mathcal{C}_j \in \mathbf{S}_*(\mathcal{A})$, while 0 is the only non-distinguished value of $\mathcal{A}$. Let $\mathcal{H}$ be the submatrix of $\mathcal{G}$ generated by $\{a,b,c\}$, in which case $h \triangleq (g{\restriction}H) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{H},\mathcal{B})$, for $g[\{a,b,c\}] = A$, while, since $\pi_j[\{a,b,c\}] = 2$ forms a subalgebra of $\mathfrak{A}$, $f \triangleq (\pi_j{\restriction}H) \in \hom(\mathcal{H},\mathcal{A}{\restriction}2)$ is surjective. Consider the following complementary (for $\sim^{\mathfrak{A}}\frac{1}{2} \in 2$) subcases:
  - $\sim^{\mathfrak{A}}\frac{1}{2} = 1$, in which case $\mathcal{B}$ is weakly $\sim$-negative, for $\sim^{\mathfrak{A}}0 = 1 \in D^{\mathcal{B}}$, and so is $\mathcal{H}$, in view of Remark 2.8(ii)(a). Then, by the following claim, $f \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{H},\mathcal{A}{\restriction}2)$, for $\mathcal{A}{\restriction}2$ is $\sim$-negative:

    Claim 6.62. *Let $\mathcal{B}$ and $\mathcal{D}$ be Σ-matrices. Suppose $\mathcal{B}$ is weakly $\sim$-negative, while $\mathcal{D}$ is consistent but not $\sim$-paraconsistent (in particular, $\sim$-negative; cf. Remark 2.8*(i)(c)*). Then, any $h \in \hom(\mathcal{B},\mathcal{D})$ is strict.*

    Proof. Take any $d \in (D \setminus D^{\mathcal{D}}) \neq \varnothing$. If, for any $b \in (B \setminus D^{\mathcal{B}})$, $h(b)$ was in $D^{\mathcal{D}}$, then, by the weak $\sim$-negativity of $\mathcal{B}$, we would have $\sim^{\mathfrak{B}} b \in D^{\mathcal{B}}$, in which case we would get $\sim^{\mathfrak{D}} h(b) = h(\sim^{\mathfrak{B}} b) \in h[D^{\mathcal{B}}] \subseteq D^{\mathcal{D}}$, and so (2.10) would not be true in $\mathcal{D}$ under $[x_0/h(b), x_1/d]$ (in particular, $\mathcal{D}$ would be $\sim$-paraconsistent). ☐

  - $\sim^{\mathfrak{A}}\frac{1}{2} = 0$, in which case $\mathcal{A}$ is $\sim$-negative, and so, by Remarks 2.6(ii), 2.8(i)(a,b) and Theorem 3.2, h is injective, for $\mathcal{A}$ is conjunctive and hereditarily simple. Then, $(h^{-1} \circ f) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{B},\mathcal{A}{\restriction}2)$, for $(h|f)(a/b) = (1/0) \in / \notin D^{\mathcal{B}|\mathcal{A}}$ and $(h|f)(c) = (\frac{1}{2}|0) \notin D^{\mathcal{B}|\mathcal{A}}$.

  Thus, anyway, by (2.14), C', being defined by $\mathcal{B}$, is defined by $\mathcal{A}{\restriction}2$, and so is $\sim$-classical, for $\mathcal{A}{\restriction}2$ is so. This, contradiction shows that $\mathcal{A}$ is truth-singular, in which case $\mathcal{B}$ is so, in view of Remark 2.8(ii)(c), for truth-singularity is clearly preserved under $\mathbf{P}$, and so $D^{\mathcal{B}} = \{a_i\}$ (in particular, by Lemma 6.60(ii), $a_i \neq \frac{1}{2}$, for $1 \neq \frac{1}{2}$). Then, $\beta^{\mathfrak{A}}_{\overline{\wedge}} \neq a_i = \Bbbk^{\mathcal{A}} = 0$, in which case, by Lemma 6.60(iii)(b)⇒(c), $\mathcal{A}$ is not classically-hereditary (in particular, is generated by 2), and so, by Lemma 6.24(ii), there is some embedding e of $\mathcal{A}$ into $\mathcal{G}$. Therefore, by Remark 2.6(ii), $e' \triangleq (e \circ g)$ is an embedding of $\mathcal{A}$ into $\mathcal{B}$, for $\mathcal{A}$ is simple, in which case it is an isomorphism from $\mathcal{A}$ onto $\mathcal{B}$, as $|A| = 3 \nleq k$, for no $k \in 3 = |B|$, and so $e'^{-1} \in \hom(\mathcal{B}, \mathcal{A})$ is strict.

In this way, in any case, there is some strict $h' \in \hom(\mathcal{B}, \mathcal{A}) \subseteq \hom(\mathfrak{A}, \mathfrak{A})$, in which case $h'(a_i) \in D^{\mathcal{A}} \not\ni h'(a_{1-i})$, for $a_i \in D^{\mathcal{B}} \not\ni a_{1-i}$, and so $h' \in \eth(\mathcal{A})$, as required. □

Then, combining Theorems 6.32(iii), 6.61 and Corollary 6.52 with Lemmas 6.59 and 6.60(ii, iii,iv), we immediately get the following two corollaries:

Corollary 6.63. *Suppose C is both $\overline{\wedge}$-conjunctive and non-$\sim$-classical, while $\mathcal{A}$ is false-/truth-singular. Then, C is self-extensional iff /either $h_{+,1-\Bbbk^{\mathcal{A}}}$ /"or $h_{-,\frac{1}{2}}$" is an endomorphism of $\mathfrak{A}$ [while $\mathfrak{A}$ is a $\overline{\wedge}$-semi-lattice with $\frac{1}{2} \leq^{\mathfrak{A}}_{\overline{\wedge}} 1$, whereas it is that with bound 0 and/iff it is that with dual bound 1 and/iff $\mathcal{A}$ is non-involutive and/iff $\mathcal{A}$ is classically-hereditary (i.e., C is $\sim$-subclassical), as well as $\mathcal{A}$ is not $\sim$-negative].*

Corollary 6.64. *Suppose $\mathcal{A}$ is both $\overline{\wedge}$-conjunctive and $\underline{\vee}$-disjunctive, while C is not $\sim$-classical. Then, C is self-extensional iff $h_{+,1-\Bbbk^{\mathcal{A}}} \in \hom(\mathfrak{A}, \mathfrak{A})$, in which case $\mathfrak{A}$ is a distributive $(\overline{\wedge}, \underline{\vee})$-lattice with zero 0 and unit 1, while $\mathcal{A}$ is neither involutive nor $\sim$-negative as well as classically-hereditary, and so C is $\sim$-subclassical.*

These immediately yield the self-extensionality of $[P]G_3^{(*)}$, for $h_{+,1-\Bbbk^{\mathcal{A}}}$ is an endomorphism of the underlying algebra of its conjunctive (disjunctive) characterisic matrix. And what is more, they immediately imply the non-self-extensionality of $[I]P^1$, for the underlying algebra of its conjunctive (disjunctive) characteristic matrix is not a semi-lattice at all {cf. Remark 6.58}. Likewise, the non-self-extensionality of the conjunctive (disjunctive) HZ {cf. Subparagraph 6.2.1.1.3} ensues from either the involutivity of its conjunctive (disjunctive) classically-hereditary characteristic matrix or the fact that the underlying algebra of this matrix, though being a distributive lattice, is not that with *both* zero 0 and unit 1. Finally, the above corollaries imply *immediately* the non-self-extensionality of $LP_{[01]}/KL_{3[01]}$, in view the involutivity of their conjunctive (disjunctive) classically-hereditary characteristic matrices, providing, as opposed to Example 4.18, a more [perhaps, the most] transparent and immediate *generic* insight into the non-self-extensionality of the latter independent from that of the former, and so into that of Łukasiewicz' finitely-valued logics [8] {cf. Example 4.17}, for these are expansions of KL_3. On the other hand, Corollary/Theorem 6.64/4.7 does not subsume Corollary/Theorem 6.63/6.61, due to existence of self-extensional conjunctive but non-disjunctive non-$\sim$-classical uniform three-valued Σ-logics with subclassical negation $\sim$, in view of:

Example 6.65. Let $\Sigma \triangleq \{\wedge, \sim\}$ and $\mathcal{A}$ the Σ-reduct of the [non-]truth-singular $\Sigma^{\supset}_{\sim,+,01}$-matrix specified in Subparagraph 6.2.1.1.2, in which case the former is both $\wedge$-conjunctive and non-$\sim$-negative, for the latter is so, and so $[P]G_3^{\wedge} \triangleq C$, being the Σ-fragment of the self-extensional [paraconsistent counterpart of] Gödel's three-valued logic $[P]G_3$ [3], is both $\wedge$-conjunctive and self-extensional as well as, by Remark 6.20 and Theorem 6.25, not $\sim$-classical. On the other hand, by induction on construction of any $\varphi \in \mathrm{Fm}^2_\Sigma$, we prove that either $\varphi^{\mathfrak{A}}(\frac{1}{2}, \frac{1}{2}) \neq \frac{1}{2}$ or there are some $a, b \in A$ such that $\max(a, b) \nleq \varphi^{\mathfrak{A}}(a, b)$. In case $\varphi = x_{0|1}$, taking $a \triangleq (0|1)$ and $b \triangleq (1|0)$, we get $\max(a, b) = 1 \nleq 0 = \varphi^{\mathfrak{A}}(a, b)$. Likewise, in case $\varphi = \sim\xi$, where $\xi \in \mathrm{Fm}^2_\Sigma$, as $(\mathrm{img} \sim^{\mathfrak{A}}) \subseteq 2 \not\ni \frac{1}{2}$, we have $\varphi^{\mathfrak{A}}(\frac{1}{2}, \frac{1}{2}) \neq \frac{1}{2}$. Finally, in case $\varphi = (\phi \wedge \psi)$, where $\phi, \psi \in \mathrm{Fm}^2_\Sigma$, if $\varphi^{\mathfrak{A}}(\frac{1}{2}, \frac{1}{2})$ is equal to $\frac{1}{2}$, then so is either $\phi^{\mathfrak{A}}(\frac{1}{2}, \frac{1}{2})$ or $\psi^{\mathfrak{A}}(\frac{1}{2}, \frac{1}{2})$, for $\mathcal{A}$ is classically-hereditary, while, if, for any $a, b \in A$, it holds that

$\max(a,b) \leqslant \varphi^{\mathfrak{A}}(a,b) = \min(\phi^{\mathfrak{A}}(a,b), \psi^{\mathfrak{A}}(a,b))$, then both $\max(a,b) \leqslant \phi^{\mathfrak{A}}(a,b)$ and $\max(a,b) \leqslant \psi^{\mathfrak{A}}(a,b)$ hold, and so the induction hypothesis completes the argument. In particular, $\max \cap A^2$ is not term-wise definable in $\mathfrak{A}$. Therefore, by Lemma 6.18 and Corollary 6.64, $[P]G_3^{\wedge}$ is not disjunctive. □

Example 6.66. Let $\Sigma \triangleq \{\wedge, \sim\}$ and $\mathcal{A}$ both truth-singular and involutive (in particular, non-$\sim$-negative) with $(a \wedge^{\mathfrak{A}} a) \triangleq a$, for all $a \in A$, as well as $(a \wedge^{\mathfrak{A}} b) \triangleq \frac{1}{2}$, for all $b \in (A \setminus \{a\})$. Then, $\mathfrak{A}$ is a $\wedge$-semi-lattice with bound $\frac{1}{2}$ and maximal elements in 2, in which case $\mathcal{A}$ is $\wedge$-conjunctive and, being involutive, is not $\sim$-negative, and so C is $\overline{\wedge}$-conjunctive and, by Remark 6.20 and Theorem 6.25, not $\sim$-classical. Moreover, $h_{-,\frac{1}{2}}$ is an endomorphism of $\mathfrak{A}$, so, by Corollary 6.63, C is self-extensional, while, by Corollary 6.64, C is not disjunctive. □

The latter example shows that the "involutive" alternative cannot be disregarded in Corollary 6.63, by which, among other things, any conjunctive self-extensional uniform three-valued non-$\sim$-classical logic with subclassical negation $\sim$ is a $\sim$-conservative term-wise definitional expansion of either of the three instances discussed above, and so is $\sim$-paraconsistent, unless its characteristic matrix is truth-singular. Likewise, by Corollary 6.64, any conjunctive $\veebar$-disjunctive self-extensional uniform three-valued non-$\sim$-classical logic with subclassical negation $\sim$ and [non-]truth-singular characteristic matrix is a $\sim$-conservative term-wise definitional expansion of $[P]G_3^*$, and so is [not] non-$\sim$-paraconsistent as well as [non-]($\veebar$, $\sim$)-paracomplete.
6.2.2.3.3. Implicative U3VLSN. We start from marking the framework of the self-extensionality of C under its being both non-$\sim$-classical and implicative:

Corollary 6.67. *Suppose $\mathcal{A}$ is $\sqsupset$-implicative. Then, C is not self-extensional, unless it is either $\sim$-paraconsistent or $\sim$-classical. In particular, C is not self-extensional, unless it is $\sim$-classical, whenever $\mathcal{A}$ is truth-singular (in particular, both ($\veebar$, $\sim$)-paracomplete and weakly $\veebar$-disjunctive).*

Proof. If $\mathcal{A}$ is both false-singular and non-$\sim$-paraconsistent, then it is $\sim$-negative. So, Remark 2.8(i)(d), Corollary 4.16 and Theorem 6.25 complete the argument. □

Theorem 6.68. *Suppose $\mathcal{A}$ is $\sqsupset$-implicative, while C is not $\sim$-classical. Then, the following are equivalent:*

(i) *C is self-extensional;*
(ii) *$h_{-,\frac{1}{2}} \in \hom(\mathfrak{A}, \mathfrak{A})$ [while $\mathfrak{A}$ is an $\sqsupset$-implicative intrinsic semi-lattice with bound $\frac{1}{2}$, whereas $\mathcal{A}$ is both false-singular and involutive as well as not classically-hereditary, and so C is not $\sim$-subclassical];*
(iii) *$\mathcal{A}_{\frac{1}{2}}$ is a [$\sim$-paraconsistent] model of C;*
(iv) *C is non-maximally $\sim$-paraconsistent.*

Proof. First, the equivalence of (iv) and the optional version of (iii) is due to Lemma 6.29(i)⇔(iv). Next, the fact that the non-optional version of (ii/iv) implies (i) is by Theorem 6.54/"4.1(vi)⇒(i) with $\mathsf{S} = \{\mathcal{A}, \mathcal{A}_{\frac{1}{2}}\}$, for $(\theta^{\mathcal{A}} \cap \theta^{\mathcal{A}_{\frac{1}{2}}}) = \Delta_A$". Further, assume the optional version of (ii) holds. Then, $h_{-,\frac{1}{2}}$ is a strict surjective homomorphism from $\mathcal{B} \triangleq \langle \mathfrak{A}, \{0, \frac{1}{2}\} \rangle$ onto $\mathcal{A}$, for this is false-singular, in view of (ii), in which case, by (2.14), $\mathcal{B}$ is a model of C, for $\mathcal{A}$ is so, and so is $\mathcal{A}_{\frac{1}{2}}$, for $\{\frac{1}{2}\} = (D^{\mathcal{A}} \cap D^{\mathcal{B}})$. Thus, the optional version of (ii) holds, for the involutivity of $\mathcal{A}$ implies the $\sim$-paraconsistency of the consistent $\mathcal{A}_{\frac{1}{2}}$. Finally, assume (i) holds. Then, by Theorem 4.9, $\mathfrak{A}$ is an $\sqsupset$-implicative intrinsic semi-lattice with bound $a \triangleq (\frac{1}{2} \sqsupset^{\mathfrak{A}} \frac{1}{2}) = (b \sqsupset^{\mathfrak{A}} b)$, for any $b \in A$, while, by Corollary 6.67, $\mathcal{A}$ is $\sim$-paraconsistent (in particular, false-singular), in which case $a \in D^{\mathcal{A}} = \{\frac{1}{2}, 1\}$, and so $a = \frac{1}{2}$ [in particular, $\sim^{\mathfrak{A}} a \in D^{\mathcal{A}}$, and so $\sim^{\mathfrak{A}} a = \frac{1}{2}$], for, otherwise, we would have $[\sim^{\mathfrak{A}}]a = 1$, in which case we would get $\sim^{\mathfrak{A}}[\sim^{\mathfrak{A}}]a = \sim^{\mathfrak{A}} 1 = 0 \notin D^{\mathcal{A}}$, and so $\mathcal{A}$ would be $\wr$-negative, where $\wr x_0 \triangleq (x_0 \sqsupset \sim[\sim](x_0 \sqsupset x_0))$ (in particular, by Corollary 6.56, C would not be self-extensional). In that case, $\mathcal{A}$ is involutive as well as not classically-hereditary, for $(0 \sqsupset^{\mathfrak{A}} 0) = a = \frac{1}{2} \notin 2 \ni 0$, while, for any $h \in \hom(\mathfrak{A}, \mathfrak{A})$, we have $h(\frac{1}{2}) = (h(\frac{1}{2}) \sqsupset^{\mathfrak{A}} h(\frac{1}{2})) = \frac{1}{2}$, so Theorems 6.32 and 6.54 end the proof. □

It is remarkable that Theorem 6.68(i)⇔(iv) appears to be opposite to Theorem 6.14. Corollary 6.64/6.63 and Theorem 6.68, in particular, "provide one more insight into ttheir context's being disjoint, in view of opposite requirements on the involitivity of characteristic matrices"/"taking Example 4.2 into account, immediately yield the following essential (mainly, due to elimination of the disjunctivity stipulation) enhancement of Theorem 6.54":

Corollary 6.69. *Suppose $\mathcal{A}$ is either implicative or conjunctive. Then, C is self-extensional iff either it is $\sim$-classical or* $(\{h_{+,1-\Bbbk^{\mathcal{A}}}, h_{-,\frac{1}{2}}\} \cap \hom(\mathfrak{A}, \mathfrak{A})) \neq \varnothing$.

Finally, we present a term-wise definitionally minimal instance of a self-extensional paraconsistent implicative U3VLSN:

Example 6.70. Let $\Sigma \triangleq \Sigma^{\supset}_{\sim}$ and $\mathcal{A}$ both false-singular and involutive with $(a \supset^{\mathfrak{A}} a) \triangleq \frac{1}{2}$ and $(a \supset^{\mathfrak{A}} b) \triangleq b$, for all $a \in A$ and all $b \in (A \setminus \{a\})$. Then, $\mathcal{A}$ is both $\sim$-paraconsistent and $\supset$-implicative. And what is more, $h_{-,\frac{1}{2}} \in \hom(\mathfrak{A}, \mathfrak{A})$. Hence, by Theorem 6.68, C is self-extensional. Now, let $\Sigma' \ni \sim$ be a signature with (possibly, secondary) binary connective $\sqsupset$, $\mathcal{A}'$ an $\sqsupset$-implicative canonical $\sim$-super-classical Σ'-matrix and C' the logic of $\mathcal{A}'$. Assume C' is self-extensional. Then, by Corollary 6.67 and Theorem 6.68, $\mathcal{A}'$ is false-singular, in which case $D^{\mathcal{A}'} = D^{\mathcal{A}}$, as well as involutive, in which case $\sim^{\mathfrak{A}'} = \sim^{\mathfrak{A}}$, while $\mathfrak{A}'$ is an $\sqsupset$-implicative intrinsic semi-lattice with bound $\frac{1}{2} = (a \sqsupset^{\mathfrak{A}[']} a)$, for any $a \in A' = A$, whereas $h \triangleq h_{-,\frac{1}{2}} \in \hom(\mathfrak{A}', \mathfrak{A}')$. Therefore, by (4.2), for all $a \in A$, $(\frac{1}{2} \sqsupset^{\mathfrak{A}'} a) = ((a \sqsupset^{\mathfrak{A}'} a) \sqsupset^{\mathfrak{A}'} a) = a$. Furthermore, by the $\sqsupset$-implicativity and false-singularity of $\mathcal{A}$, for each $b \in D^{\mathcal{A}}$, $(b \sqsupset^{\mathfrak{A}'} 0) = 0$, and so $(h(b) \sqsupset^{\mathfrak{A}'} 1) = h(0) = 1$. Likewise, $(0 \sqsupset^{\mathfrak{A}'} b) \in D^{\mathcal{A}}$, in which case $(0 \sqsupset^{\mathfrak{A}'} \frac{1}{2}) = \frac{1}{2}$, for, otherwise, $D^{\mathcal{A}} \ni (1 \sqsupset^{\mathfrak{A}'} \frac{1}{2}) = h(1) = 0 \notin D^{\mathcal{A}}$, while $(0 \sqsupset^{\mathfrak{A}'} 1) = 1$, for, otherwise, $D^{\mathcal{A}} \not\ni (1 \sqsupset^{\mathfrak{A}'} 0) = h(\frac{1}{2}) = \frac{1}{2} \in D^{\mathcal{A}}$, and so $(1 \sqsupset^{\mathfrak{A}'} \frac{1}{2}) = h(\frac{1}{2}) = \frac{1}{2}$. In this way, $\sqsupset^{\mathfrak{A}'} = \supset^{\mathfrak{A}}$. Thus, C' is a $\sim$-conservative term-wise definitional expansion of C. □

7. Conclusions

Aside from quite useful general results and their equally illustrative generic applications (sometimes, even multiple ones providing different insights, and so demonstrating the whole power of universal tools elaborated here) to infinite classes of particular logics, the incompatibility of the self-extensionality of either implicative or both conjunctive and disjunctive finitely-valued logics with unitary equality determinant and the algebraizability (in the sense of [17, 16]) of two-side sequent calculi (associated with such logics according to [18]), discovered here, looks quite remarkable, especially due to its providing a new insight into the non-"self-extensinality of"/"algebraizability of sequent calculi associated with" certain logics of such a kind proved originally *ad hoc*, and so justifying the thesis of the first paragraph of Section 1. Likewise, equivalence of structural completeness and maximal paracompleteness of both uniform four-valued expansions of Belnap's logic and weakly disjunctive paracomplete U3VLSN as well as equally interesting connections between maximal paraconsistency and implicativity/self-extensionality of self-extensional/implicative uniform "four-valued expansions of Belnap's logic"/"three-valued logics with subclassical negation" deserve a particular emphasis within the context of General Logic. And what is more, Subsection 6.2 constitutes foundations of an algebraic theory of U3VLSN. In this connection, taking Theorem 6.61 into account, the most acute problem remaining still open is marking the framework of elimination of disjuctivity stipulation in Theorem 4.7.

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