Exact Separation of Long- and Short-Period Effects in the Computation of Mean Elements of Artificial Satellite Theory

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Abstract

It is well known that mean elements obtained by canonical perturbation theory only agree with the average dynamics of the osculating orbit up to first order effects. While this fact does not necessarily compromise the accuracy of corresponding perturbation solutions, the loose use of the terminology "mean elements" in artificial satellite theory may obscure the understanding of the variety of available solutions in the literature, and thus make the implementation of additional patches to increase their performance ambiguous. After briefly reviewing the topic, the purely periodic, non-canonical, mean to osculating transformation that yields the exact separation between short- and long-period variations is computed for the main problem of artificial satellite theory up to the second order of the zonal harmonic of the second degree. It is also shown that this kind of non-canonical solution confines the long-period oscillations of the semimajor axis in the mean variation equations.

1 Introduction

The decomposition of orbital motion into secular, long- and short-period effects allowed astronomers of the 18th and 19th centuries to better understand the dynamics of celestial objects, and, therefore, make reasonably accurate predictions of their motions. Analogous techniques were successfully applied to the prediction of orbits of artificial satellites since the beginning of the space era [1, 2, 3].

The amplitudes of the fast, short-period fluctuations that modulate the long-term dynamics are commonly small, and are conveniently removed by averaging techniques in order to more easily predict the slow variations of the "mean" orbital elements. The averaging is mathematically supported by a transformation from osculating to mean

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variables. However, different changes of variables can be used to remove the short-period effects, a fact that makes the definition of mean elements non-unique, and even sometimes confusing [4]. Ideally, the transformation should remove only the short-period effects, so that the resulting mean elements capture all the effects that drive the long-term dynamics. This removal is easily done when the mean anomaly appears explicitly in the variation equations. But this is not the case of artificial satellite theory, where making the mean anomaly explicit requires carrying out expansions of the elliptic motion [5]. These expansions constrain the range of the eccentricities to which the solution applies, and may result in very long trigonometric series [6].

High-eccentricity orbits provide a variety of opportunities for satellite missions, and, therefore the removal of the mean anomaly in closed form is clearly wanted. It is also feasible, at least for usual perturbation models, yet the closed form solution may hide the simultaneous removal of additional long-period effects as a side effect of the procedure [7]. This unintended removal of part of the long-period terms from the mean elements dynamics is not of worry for analytical or semi-analytical ephemeris computation. However, it can certainly be a concern for some space geodesy applications, which may require the exact separation of the short-period fluctuations from the long-term dynamics in order to get mean elements that strictly adhere to the average evolution of the osculating elements [8].

The exact decomposition of the orbital motion into pure short- and long-period effects is readily achieved with different perturbation techniques by the proper choice of the arbitrary integration functions of the slow variables which arise in the analytical computation of the mean to osculating transformation [9, 10, 11]. However, this is true only for the linear part of the mean to osculating transformation, beyond which the widespread canonical perturbation methods are no longer useful [8, 12]. While first order perturbation theories are in the roots of useful operational software dealing with real perturbation models, second order effects of the Earth's zonal harmonic of the second degree cannot be ignored in the implementation of the perturbation solution [13, 14, 15, 16, 17]. Still, the exact separation of the short-period part of these effects is rarely achieved, and, to our knowledge, the literature lacks of explicit expressions for the corresponding second order terms of the mean to osculating transformation, with the exception of simple toy models not suitable for dealing with the real dynamics [18].

In this paper we fill the gap and, for the main problem of artificial satellite theory, compute the second-order terms of the mean to osculating transformation that is the pure periodic in the mean anomaly in closed form of the eccentricity. As expected, this mean to osculating transformation differs from the standard results only in the long-period terms that affect the latter. While the whole long-period effects of the dynamics are now moved to the mean frequencies, the changes only affect to the mean variation of the mean anomaly for a second-order theory. We also compute the third-order components of the mean variations, which can be done based only on the knowledge of the second-order transformation. These higher-order terms show the non-vanishing character of the mean variation of the semimajor axis. This feature is in clear contrast with the case of Hamiltonian perturbation methods, in which the mean variation of the semimajor axis vanishes at any order, and its long-period oscillations are only recovered through the mean to osculating transformation.

The computation of the third-order terms of the perturbation theory confronts the closed form integration of non-trivial functions depending on the equation of the center, which, therefore, loose the trigonometric character. Still, we will see that the solution of the required integrals can be approached by standard integration by parts. This standard procedure reduces the problem to the integration of known functions of the elliptic motion, whose solutions are either trivial or have been previously reported in the literature (refer to [19, 20, 21, 22] and references therein).

In spite of the searched mean to osculating transformation is non-canonical by nature, the new perturbation theory has been computed in Delaunay canonical action-angle variables for simplicity, on the one hand, and with the aim of easing comparison with previous results in the literature, on the other hand. Obtaining the perturbation solution in a different set of variables, as for instance non-singular ones, may requires the re-computation of the perturbation theory following analogous steps as the ones described here if we want to preserve the pure periodic character of the mean to osculating transformation of all the chosen variables.

2 The main problem

Constraining to the main part, the gravitational potential of the Earth at a point defined by the spherical coordinates (r, φ, λ), for geocentric distance, latitude, and longitude, respectively, is

$$\mathcal{V} = -\frac{\mu}{r} + \frac{\mu}{r} J_2 \frac{R_\oplus^2}{r^2} \frac{1}{2}(3\sin^2\varphi - 1), \tag{1}$$

where the physical parameters μ, $R_\oplus$, and J_2, denote the gravitational parameter, equatorial radius, and oblateness coefficient, respectively. The dynamical model represented by Eq. (1) is customarily known as the main problem of artificial satellite theory [2]. Because J_2 is commonly very small, the solution to the main problem can be approached as a perturbation of the integrable, Keplerian potential $\mathcal{V}_{\text{Kepler}} = -\mu/r$.

Finding approximate perturbation solutions to the main problem is more suitably approached in Delaunay canonical variables (ℓ, g, h, L, G, H), whose definition is commonly expressed in terms of the more familiar, non-canonical Keplerian variables as $\ell = M$, $g = \omega$, $h = \Omega$, $L = \sqrt{\mu a}$, $G = L(1-e^2)^{1/2}$, $H = G\cos i$, where M denotes the mean anomaly, ω the argument of the perigee, Ω the right ascension of the ascending node, a the semimajor axis, e the eccentricity, and i the inclination.

The implicit dependence of the true anomaly on the mean one complicates the closed-form solution of the variation equations in Delaunay variables. More precisely, the closed-form solution of the integrals that appear in the perturbation approach is achieved after making a change of the independent variable based on the preservation of the angular momentum of the Keplerian motion $G = r^2 \mathrm{d}f/\mathrm{d}t$. Then, in preparation of the perturbation approach, the variation equations of the Delaunay osculating elements are conveniently arranged in the style of in [23]. That is,

$$\frac{\mathrm{d}\ell}{\mathrm{d}t} = n + n\frac{J_2}{4}\frac{R_\oplus^2}{r^2}\frac{3}{\eta^2}\Big[-(3s^2-2)(1+e\cos f) + \frac{3s^2-2}{4e}\sum_{j=1}^{3}\omega_{0,j}\cos jf$$

$$
- \frac{1}{8e} \sum_{j=-1}^{5} \sum_{k=0}^{j \bmod 2} M^{*}_{j,k} e^{j \bmod 2 - 2k + 1} \cos(jf + 2\omega) \Big], \tag{2}
$$

$$
\frac{\mathrm{d}g}{\mathrm{d}t} = n \frac{J_2}{4} \frac{R_\oplus^2}{r^2} \frac{3}{\eta^3} \Big[-(5s^2 - 4)(1 + e\cos f) - \frac{3s^2 - 2}{4e} \sum_{j=1}^{3} \omega_{0,j} \cos jf
$$

$$
+ \frac{1}{8e} \sum_{j=-1}^{5} \sum_{k=0}^{j \bmod 2} \omega^{*}_{j,k} e^{j \bmod 2 - 2k + 1} \cos(jf + 2\omega) \Big], \tag{3}
$$

$$
\frac{\mathrm{d}h}{\mathrm{d}t} = n \frac{J_2}{4} \frac{R_\oplus^2}{r^2} \frac{c}{\eta^3} \Big[-6(1 + e\cos f) + \sum_{j=1}^{3} \Omega_{1,j} \cos(jf + 2\omega) \Big], \tag{4}
$$

$$
\frac{\mathrm{d}L}{\mathrm{d}t} = n \frac{J_2}{4} \frac{R_\oplus^2}{r^2} \frac{3L}{8\eta^5} \Big[2(3s^2 - 2) \sum_{j=1}^{3} L_{0,j} \sin jf - s^2 \sum_{j=-1}^{5} L_{1,j} \sin(jf + 2\omega) \Big], \tag{5}
$$

$$
\frac{\mathrm{d}G}{\mathrm{d}t} = -n \frac{J_2}{4} \frac{R_\oplus^2}{r^2} \frac{G}{\eta^3} s^2 \sum_{j=1}^{3} \Omega_{1,j} \sin(jf + 2\omega), \tag{6}
$$

$$
\frac{\mathrm{d}H}{\mathrm{d}t} = 0, \tag{7}
$$

where $n = (\mu/a^3)^{1/2}$ is the Keplerian mean motion, $r = p/(1 + e\cos f)$, $p = a\eta^2$, $\eta = (1-e^2)^{1/2}$, the true anomaly f is an implicit function of the mean anomaly and the eccentricity, which requires the solution of Kepler's equation, $s = \sin i$, $c = \cos i$, and the Keplerian elements are functions of the Delaunay variables given by its definition. The eccentricity polynomials $L_{i,j}$, $\Omega_{i,j}$, $\omega_{i,j}$ are given in Table 1 for non-vanishing values, and the inclination polynomials $\omega^{*}_{i,j}$, and $M^{*}_{i,j}$ are presented in Table 2.

Table 1: Eccentricity polynomials in Eqs. (2)–(6).

i,j	0,1	0,2	0,3	1,−1	1,1	1,2	1,3	1,4	1,5
$L_{i,j}$	$4e + e^3$	$12e^2$	$3e^3$	$-3e^3$	$9e^3 + 36e$	$72e^2 + 48$	$27e^3 + 108e$	$72e^2$	$15e^3$
$\Omega_{i,j}$	–	–	–	–	$3e$	6	$3e$	–	–
$\omega_{i,j}$	$12 - 3e^2$	$12e$	$3e^2$	–	–	–	–	–	–

Table 2: Inclination polynomials $Q_{i,j}$ in Eqs. (2) and (3).

$Q_{i,j}$	−1,0	1,0	1,1	2,0	3,0	3,1	4,0	5,0
ω^*	s^2	$15s^2 - 8$	$-4s^2$	$40s^2 - 16$	$19s^2 - 8$	$28s^2$	$24s^2$	$5s^2$
M^*	s^2	$-17s^2$	$-4s^2$	$-24s^2$	$-13s^2$	$28s^2$	$24s^2$	$5s^2$

The Delaunay variables share the same deficiencies as the Keplerian elements, and hence the eccentricity in denominators of Eqs. (2) and (3). The singularity for circular orbits is non-essential, and disappears when using non-singular variables [24, 6, 25].

The preservation of H stemming from the vanishing of Eq. (7) decouples the variation of the right ascension of the ascending node in Eq. (4) of the J_2 problem from the reduced dynamics given by Eqs. (2)–(3) and (5)–(6). Therefore, the qualitative aspects of the decoupled, two degrees of freedom dynamics can be approached with the straightforward computation of Poincaré surfaces of sections, which clearly show the existence of chaotic regions either in the cylindrical map [26, 27] or in the equivalent dynamics in the orbital plane, cf. §5.5 of [28]. While the non-integrability of the J_2 problem is also supported by analytical results [29, 30], it happens that the size of the chaotic regions is very small for the particular value of the Earth's J_2 coefficient [31]. Therefore, in spite of the closed form solution to Eqs. (2)–(7) does not exist, searching for analytical approximations to the main problem dynamics makes sense. In fact, analytical perturbation solutions to the artificial satellite problem may keep machine precision over long time spans in the regions in which these kinds of solutions exist [32]. Alternatively, solutions based on the classical Picard iterations scheme can be always computed for arbitrary initial conditions, yet their validity may be restricted to shorter times [23].

3 Non-canonical perturbation solution

A semi-analytic perturbation theory consists of two fundamental blocks. Namely, the mean to osculating transformation and the mean variation equations. The first is given in the form of truncated Taylor series

$$x_j = x'_j + \sum_{i=1}^{m} \frac{\epsilon^i}{i!} \Delta_{j,i}(x'_k) + \mathcal{O}(\epsilon^{m+1}), \tag{8}$$

where $j, k = 1, \ldots, 6$, ϵ is a small parameter, either physical or formal, and primes denote mean elements. The functions $\Delta_{j,i}$ of the mean elements are commonly trigonometric for perturbed Keplerian motion, and may comprise non-periodic as well as short- and long-period terms. The mean variation equations have also de form of truncated series

$$\frac{\mathrm{d}x'_j}{\mathrm{d}t} = \sum_{i=0}^{q} \frac{\epsilon^i}{i!} F_{j,i}(x'_k) + \mathcal{O}(\epsilon^{q+1}), \tag{9}$$

where $j = 1, \ldots, 6$, $k = 2, \ldots, 6$, and now, after truncation, the fast variable, say x'_1, is absent in the right side of the equations. In consequence, the differential system in the prime variables can be numerically integrated with large step sizes. The mean variation of the fast variable is thus decoupled, and sometimes ignored in the integration. Remark that the computation of the m order of the transformation in Eq. (8) frequently allows to obtain the mean frequencies in Eq. (9) up to $q = m + 1$. Therefore, an m order theory is hereafter understood as comprising the mean to osculating transformation up to the order m, and the variation equations up to the order $m + 1$.

Then, the semi-analytic propagation of an orbit for given initial conditions proceeds in three different steps. To wit, i) the initial conditions of the osculating orbit are first transformed into corresponding initial conditions in mean variables; ii) these initial mean variables feed the mean variation equations, which are then integrated numerically for a given time; and iii) the osculating elements are recovered at desired outputs of the numerical integration using the mean to osculating transformation in order to obtain the ephemeris. Depending on the engineering application the last step is not always carried out, and hence the importance of the completeness of the mean elements.

Step i) requires to invert Eq. (8), namely

$$x'_j = x_j + \sum_{i=1}^{m} \frac{\epsilon^i}{i!} \Delta'_{j,i}(x_k) + \mathcal{O}(\epsilon^{m+1}), \quad j,k = 1,2,\ldots,6, \tag{10}$$

which can be approached in different ways [33, 34, 35, 36]. Still, since the truncation of the transformation is one order lower than that of the mean variations, the initialization process will introduce $\mathcal{O}(\epsilon^{m+1})$ errors that are critical for the computation of the mean frequencies. In particular, inaccuracies in the initialization of the mean motion in mean elements by this cause unavoidably yield the abnormal propagation of in-track errors [37, 38]. Keeping in-track accuracy requires the proper initialization of the mean motion. This operation is routinely carried out either by a fit of the mean frequencies to a preliminary numerical integration of the osculating variations for just a few orbits, or by the calibration of the semimajor axis by other means [39]. Remarkably, the calibration process is unnecessary for perturbation approaches in the extended phase space [40, 41, 42].

The computation of Eqs. (8) and (9) for a given perturbation model can be achieved with different perturbation methods [43]. Here we used the method of Lie transforms [44, 45], which has the advantage of providing the inverse transformation of Eq. (8) explicitly, and can be implemented up to arbitrary orders with powerful recursive algorithms [45, 46, 47].

At the first order, the components $F_{j,1}$ of the mean variations in Eq. (9) are

$$F_{1,1} = -n\frac{R_\oplus^2}{p^2}\frac{3}{4}(3s^2-2)\eta, \tag{11}$$

$$F_{2,1} = -n\frac{R_\oplus^2}{p^2}\frac{3}{4}(5s^2-4), \tag{12}$$

$$F_{3,1} = -n\frac{R_\oplus^2}{p^2}\frac{3}{2}c, \tag{13}$$

$$F_{4,1} = 0, \tag{14}$$

$$F_{5,1} = 0, \tag{15}$$

a standard result that has been repeatedly reported in the literature [2, 48, 49]. The corresponding elements of the mean to osculating transformation Eq. (8) are

$$\begin{aligned}\Delta_1\ell = \frac{R_\oplus^2}{p^2}\frac{\eta}{32e}\Big\{&(6s^2-4)\big[e^2\sin 3f - 3(e^2-4)\sin f + 6e\sin 2f\big]\\ &- s^2\big[3e^2\sin(f-2g) + 2\frac{4\eta^3-\eta^2-18\eta-9}{(\eta+1)^2}e\sin 2g + 3e^2\sin(5f+2g)\\ &- 3(5e^2+4)\sin(f+2g) - 3(5e^2+4)\sin(f+2g)\\ &- (e^2-28)\sin(3f+2g) + 18e\sin(4f+2g)\big]\Big\}\end{aligned} \tag{16}$$

$$
\begin{aligned}
\Delta_1 L = L\frac{R_\oplus^2}{p^2}\frac{1}{32\eta^2}\Big\{&(6s^2-4)\big[2(\eta-1)(2\eta^2+5\eta+5)-e^3\cos 3f-6e^2\cos 2f\\
&+3e(\eta^2-5)\cos f\big]+3s^2\big[e^3\cos(f-2g)+e^3\cos(5f+2g)\\
&+6e^2\cos(4f+2g)+6e^2\cos 2g-3e(\eta^2-5)\cos(f+2g)\\
&-3e(\eta^2-5)\cos(3f+2g)-4(3\eta^2-5)\cos(2f+2g)\big]\Big\}
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
\Delta_1 g =\ &-\frac{R_\oplus^2}{p^2}\frac{3}{4}(5s^2-4)\phi+\frac{R_\oplus^2}{p^2}\frac{1}{32e}\Big\{3e^2s^2\sin(f-2g)+3e^2s^2\sin(5f+2g)\\
&+3\big[e^2(15s^2-8)-4s^2\big]\sin(f+2g)+\big[e^2(19s^2-8)+28s^2\big]\sin(3f+2g)\\
&-2e^2(3s^2-2)\sin 3f-6\big[e^2(17s^2-14)+4(3s^2-2)\big]\sin f\\
&+12e(5s^2-2)\sin(2f+2g)+18es^2\sin(4f+2g)-12e(3s^2-2)\sin 2f\\
&-\frac{2}{(\eta+1)^2}\big[8\eta^3(2s^2-1)+\eta^2(11s^2-4)-(2\eta+1)(s^2-4)\big]e\sin 2g\Big\}
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
\Delta_1 G = G\frac{R_\oplus^2}{p^2}\frac{s^2}{4}\big[&3e\cos(f+2g)+e\cos(3f+2g)+3\cos(2f+2g)\\
&+\frac{2\eta+1}{(\eta+1)^2}e^2\cos 2g\big]
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
\Delta_1 h =\ &-\frac{R_\oplus^2}{p^2}\frac{3}{2}c\phi+\frac{R_\oplus^2}{p^2}\frac{1}{4}c\Big\{-6e\sin f+\frac{2\eta+1}{(\eta+1)^2}e^2\sin 2g+3e\sin(f+2g)\\
&+3\sin(2f+2g)+e\sin(3f+2g)\Big\}
\end{aligned}
\tag{20}
$$

where $\phi = f - \ell$ denotes the equation of the center, which, while non-trigonometric, is 2π-periodic in the mean anomaly [5]. Note that the appearance of long-period terms with only argument $2g$ is in fact needed to cancel other long-period terms that remain hidden due to their dependence on the true anomaly instead of the mean one.

The periodic corrections in Eqs. (16)–(20) are valid for both the direct and inverse transformations. However, they must be evaluated in mean, prime Delaunay variables in the mean to osculating transformation Eq. (8), whereas they must be evaluated in osculating, non-primed Delaunay variables, in the inverse, osculating to mean transformation Eq. (10), which besides must have the opposite signs of Eqs. (16)–(20). That is, $\Delta'_{j,1}(x_k) \equiv -\Delta_{j,1}(x'_k = x_k)$.

The computation of the second order terms $F_{j,2}$ of the mean variations (9) only involves partial differentiation and averaging operations. Non-trivial quadratures related to different combinations of the equation of the center with trigonometric functions of the true anomaly are readily solved with the help of the general formulas in [20]. We obtain,

$$
\begin{aligned}
F_{1,2} = n\frac{R_\oplus^4}{p^4}\frac{3}{512}\frac{1}{\eta}\Big\{&2\big[\eta^4(403s^4+40s^2-184)+64\eta^3(3s^2-2)^2-30\eta^2(53s^4-8\\
&-8s^2)+105(27s^4-24s^2+8)\big]-16\big[\eta^4(147s^2-130)+56\eta^3(5s^2-4)\\
&-6\eta^2(111s^2-86)+40\eta(5s^2-4)+8\frac{3\eta+4}{(1+\eta)^2}(5s^2-4)+29s^2+2\big]s^2
\end{aligned}
$$

$$\times \cos 2g + 27e^4 s^4 \cos 4g\Big\}, \tag{21}$$

$$\begin{aligned} F_{2,2} = {}& n\frac{R_\oplus^4}{p^4}\frac{3}{64}\Big\{\eta^2(45s^4 + 36s^2 - 56) + 24\eta(3s^2 - 2)(5s^2 - 4) + 5(77s^4 + 88 \\ & - 172s^2) - \frac{2}{(\eta+1)^2}\big[\eta^4(135s^4 - 158s^2 + 28) + 2\eta^3(335s^4 - 366s^2 + 60) \\ & + 2\eta^2(55s^4 - 66s^2 + 16) - 10\eta(77s^4 - 82s^2 + 12) - 5(77s^4 - 82s^2 \\ & + 12)\big]\cos 2g\Big\}, \end{aligned} \tag{22}$$

$$\begin{aligned} F_{3,2} = {}& cn\frac{R_\oplus^4}{p^4}\frac{3}{16}\{\eta^2(5s^2 + 4) + 12\eta(3s^2 - 2) + 5s(7s^2 - 8) \\ & + 2\big[4(5s^2 - 2)\frac{2\eta + 1}{(1+\eta)^2} + 15s^2 - 7\big]e^2\cos 2g\}, \end{aligned} \tag{23}$$

$$F_{4,2} = 0, \tag{24}$$

$$F_{5,2} = -Gn\frac{R_\oplus^4}{p^4}\frac{3}{16}\{\big[4(5s^2 - 4)\frac{2\eta + 1}{(\eta+1)^2} + 15s^2 - 14\big]e^2 s^2 \sin 2g\}. \tag{25}$$

The computation of the second order terms of the mean to osculating transformation in closed form requires the solution of more sophisticated indefinite integrals involving the equation of the center. More precisely, we find integrands of the form $(p/r)^2\phi^m \mathrm{trig}(mf + \xi)$, where trig applies to both sine and cosine functions, and ξ stands for some multiple of the slow varying angles. As detailed in Appendix A, standard integration by parts reduces these integrals to the case of known functions. On the other hand, because the corrections $\Delta_{j,2}$ and $\Delta'_{j,2}$ in Eqs. (8) and (10), respectively, comprise long series, we only provide the second order term of the osculating-to-mean transformation of L, which should be used in the first-order theory to calibrate the semimajor axis in order to avoid the abnormal growth of in-track errors. We obtained,

$$\begin{aligned} \Delta'_2 L = {}& L\frac{R_\oplus^4}{p^4}\frac{3}{2048\eta^4(1+\eta)^2}\Bigg[\sum_{i=0}^{9}\sum_{k=0}^{1}\sum_{j=-4k}^{6+2k}\eta^i s^{2k} e^{j \bmod 2} q_{i,j,k}\cos(jf + 2kg) \\ & + s^4(1+\eta)^2\sum_{j=-2}^{10} e^{(j \bmod 2)} Q_j \cos(jf + 4g)\Bigg], \end{aligned} \tag{26}$$

where the eccentricity polynomials Q_j are presented in Table 3, in which we avoid repetition by noting that $Q_j = Q_{8-j}$ save for $j = 0$. The non-null inclination polynomials $q_{i,j,k}$ are presented in Tables 4 and 5. Because the osculating-to-mean transformation only needs to be evaluated once, in the initialization process of the semi-analytic integration, we do not make claims about the efficiency in the evaluation of Eq. (26), which will probably improved replacing some of the trigonometric functions by different powers of the radius [3].

The second order theory includes also the third order terms $F_{j,3}$ of the mean variations (9). The printed expressions are omitted for brevity, as we did with the second order transformation, save for the one corresponding to the Delaunay action, which

Table 3: Eccentricity polynomials Q_j in (26)

$_0: 18e^4(2\eta^3-3\eta^2+33)$
$_4: -36(5\eta^6-105\eta^4+315\eta^2-231)$
$_5: 216(5\eta^4-30\eta^2+33)$
$_6: -135(1+\eta)(\eta^5-\eta^4-18\eta^3+18\eta^2+33\eta-33)$
$_7: 180(1+\eta)(3\eta^3-3\eta^2-11\eta+11)$
$_8: -54e^4(\eta^2-11)$
$_9: 108e^4$
$_{10}: 9e^6$

Table 4: Inclination polynomials $q_{i,j,0}$ in (26); $\rho_0=27s^4-24s^2+8$

$_{0,0}: 924\rho_0$	$_{5,1}: 480\rho_0$	$_{4,3}: -440\rho_0$	$_{3,5}: -96\rho_0$
$_{1,0}: 1848\rho_0$	$_{6,1}: 240\rho_0$	$_{5,3}: 240\rho_0$	$_{4,5}: -24\rho_0$
$_{2,0}: -336\rho_0$	$_{0,2}: 990\rho_0$	$_{6,3}: 120\rho_0$	$_{5,5}: 48\rho_0$
$_{3,0}: -2240\rho_0$	$_{1,2}: 1980\rho_0$	$_{0,4}: 132\rho_0$	$_{6,5}: 24\rho_0$
$_{4,0}: -280\rho_0$	$_{2,2}: -540\rho_0$	$_{1,4}: 264\rho_0$	$_{0,6}: 2\rho_0$
$_{5,0}: 80(325s^4-328s^2+120)$	$_{3,2}: -3060\rho_0$	$_{2,4}: -144\rho_0$	$_{1,6}: 4\rho_0$
$_{6,0}: 80(29s^4-104s^2+56)$	$_{4,2}: -960\rho_0$	$_{3,4}: -552\rho_0$	$_{2,6}: -4\rho_0$
$_{7,0}: -256(17s^4-6s^2)$	$_{5,2}: 1140\rho_0$	$_{4,4}: -120\rho_0$	$_{3,6}: -12\rho_0$
$_{8,0}: 4(349s^4+88s^2-200)$	$_{6,2}: 540\rho_0$	$_{5,4}: 312\rho_0$	$_{5,6}: 12\rho_0$
$_{9,0}: 8(121s^4-8s^2-40)$	$_{7,2}: -60\rho_0$	$_{6,4}: 144\rho_0$	$_{6,6}: 4\rho_0$
$_{0,1}: 1584\rho_0$	$_{8,2}: -30\rho_0$	$_{7,4}: -24\rho_0$	$_{7,6}: -4\rho_0$
$_{1,1}: 3168\rho_0$	$_{0,3}: 440\rho_0$	$_{8,4}: -12\rho_0$	$_{8,6}: -2\rho_0$
$_{2,1}: 144\rho_0$	$_{1,3}: 880\rho_0$	$_{0,5}: 24\rho_0$	
$_{3,1}: -2880\rho_0$	$_{2,3}: -120\rho_0$	$_{1,5}: 48\rho_0$	
$_{4,1}: -1200\rho_0$	$_{3,3}: -1120\rho_0$	$_{2,5}: -24\rho_0$	

does not vanish for the first time. In particular

$$F_{4,3}=Ln\frac{R_\oplus^6}{p^6}\frac{27}{512\eta}(5s^2-4)\Big\{8\frac{1-\eta}{1+\eta}\big[4\eta^4(6s^2-5)+8\eta^3(11s^2-9)+\eta^2(6-19s^2)-21(2\eta+1)(3s^2-2)\big]s^2\sin 2g+9e^4s^4\sin 4g\Big\}.$$

Confining the long-period terms of the Delaunay action, or, equivalently, the semimajor axis, in the mean variations makes an important difference with respect to traditional solutions of perturbed Keplerian motion. On the other hand, this displacement of the long-period oscillations of the semimajor axis from the mean-to-osculating transformation to the mean variations may affect the accuracy of long-term propagations [18].

Table 5: Inclination polynomials $q_{i,j,1}$ in (26); $\rho_1 = 3s^2 - 2$

$_{0,-4}: -12\rho_1$	$_{3,-1}: 6720\rho_1$	$_{5,2}: -96(355s^2 - 242)$	$_{6,5}: -720\rho_1$
$_{1,-4}: -24\rho_1$	$_{4,-1}: 2640\rho_1$	$_{6,2}: -192(115s^2 - 82)$	$_{0,6}: -792\rho_1$
$_{2,-4}: 24\rho_1$	$_{5,-1}: -1440\rho_1$	$_{7,2}: -96(25s^2 - 22)$	$_{1,6}: -1584\rho_1$
$_{3,-4}: 72\rho_1$	$_{6,-1}: -720\rho_1$	$_{8,2}: 240\rho_1$	$_{2,6}: 864\rho_1$
$_{5,-4}: -72\rho_1$	$_{0,0}: -5940\rho_1$	$_{0,3}: -9504\rho_1$	$_{3,6}: 3312\rho_1$
$_{6,-4}: -24\rho_1$	$_{1,0}: -11880\rho_1$	$_{1,3}: -19008\rho_1$	$_{4,6}: 720\rho_1$
$_{7,-4}: 24\rho_1$	$_{2,0}: 3240\rho_1$	$_{2,3}: -864\rho_1$	$_{5,6}: -1872\rho_1$
$_{8,-4}: 12\rho_1$	$_{3,0}: 17016\rho_1$	$_{3,3}: 17280\rho_1$	$_{6,6}: -864\rho_1$
$_{0,-3}: -144\rho_1$	$_{4,0}: 3072\rho_1$	$_{4,3}: 7200\rho_1$	$_{7,6}: 144\rho_1$
$_{1,-3}: -288\rho_1$	$_{5,0}: -8(2093s^2 - 1310)$	$_{5,3}: -64(155s^2 - 106)$	$_{8,6}: 72\rho_1$
$_{2,-3}: 144\rho_1$	$_{6,0}: 8(737s^2 - 662)$	$_{6,3}: -32(215s^2 - 154)$	$_{0,7}: -144\rho_1$
$_{3,-3}: 576\rho_1$	$_{7,0}: 8(479s^2 - 298)$	$_{7,3}: -256(5s^2 - 4)$	$_{1,7}: -288\rho_1$
$_{4,-3}: 144\rho_1$	$_{8,0}: -4(1753s^2 - 1510)$	$_{0,4}: -5940\rho_1$	$_{2,7}: 144\rho_1$
$_{5,-3}: -288\rho_1$	$_{9,0}: -64(39s^2 - 34)$	$_{1,4}: -11880\rho_1$	$_{3,7}: 576\rho_1$
$_{6,-3}: -144\rho_1$	$_{0,1}: -9504\rho_1$	$_{2,4}: 3240\rho_1$	$_{4,7}: 144\rho_1$
$_{0,-2}: -792\rho_1$	$_{1,1}: -19008\rho_1$	$_{3,4}: 18360\rho_1$	$_{5,7}: -288\rho_1$
$_{1,-2}: -1584\rho_1$	$_{2,1}: -864\rho_1$	$_{4,4}: 5760\rho_1$	$_{6,7}: -144\rho_1$
$_{2,-2}: 864\rho_1$	$_{3,1}: 17280\rho_1$	$_{5,4}: -6840\rho_1$	$_{0,8}: -12\rho_1$
$_{3,-2}: 3312\rho_1$	$_{4,1}: 7200\rho_1$	$_{6,4}: -3240\rho_1$	$_{1,8}: -24\rho_1$
$_{4,-2}: 720\rho_1$	$_{5,1}: -192(65s^2 - 46)$	$_{7,4}: 360\rho_1$	$_{2,8}: 24\rho_1$
$_{5,-2}: -1872\rho_1$	$_{6,1}: -96(125s^2 - 94)$	$_{8,4}: 180\rho_1$	$_{3,8}: 72\rho_1$
$_{6,-2}: -864\rho_1$	$_{7,1}: -768(5s^2 - 4)$	$_{0,5}: -2640\rho_1$	$_{5,8}: -72\rho_1$
$_{7,-2}: 144\rho_1$	$_{0,2}: -11088\rho_1$	$_{1,5}: -5280\rho_1$	$_{6,8}: -24\rho_1$
$_{8,-2}: 72\rho_1$	$_{1,2}: -22176\rho_1$	$_{2,5}: 720\rho_1$	$_{7,8}: 24\rho_1$
$_{0,-1}: -2640\rho_1$	$_{2,2}: 4032\rho_1$	$_{3,5}: 6720\rho_1$	$_{8,8}: 12\rho_1$
$_{1,-1}: -5280\rho_1$	$_{3,2}: 30240\rho_1$	$_{4,5}: 2640\rho_1$	
$_{2,-1}: 720\rho_1$	$_{4,2}: 10080\rho_1$	$_{5,5}: -1440\rho_1$	

4 The canonical case

For comparison, we also computed a perturbation theory in which the integration functions of the slow variables are selected so that they make the generator purely periodic [18]. Up to the second order of the mean to osculating transformation we checked that the transformation is canonical by the standard computation of the symplectic matrix. Therefore, this perturbation theory provides the same results as a perturbation solution obtained by canonical methods, assumed, of course, that the arbitrary integration functions of the solution are fixed with the same criterion.

The differences of this theory with respect to the non-canonical case start at the second order. Now, the second order terms of the mean to osculating transformation are affected of long-period terms. In particular, the second order term of inverse correction to the Delaunay action, Eq. (26), must be supplemented with the addition of

$$\begin{aligned}\delta_2' L = & - L\frac{R_\oplus^4}{p^4}\frac{3}{512\eta}\Big\{2\big[\eta^4(101s^4-40s^2-8)-10\eta^2(67s^4-40s^2+8) \\ & +35(27s^4-24s^2+8)\big]+16\frac{1-\eta}{1+\eta}\big[4\eta^4(6s^2-5)+8\eta^3(11s^2-9) \\ & +\eta^2(6-19s^2)-21(2\eta+1)(3s^2-2)\big]s^2\cos 2g+9e^4s^4\cos 4g\Big\}. \qquad (27)\end{aligned}$$

Regarding the mean frequencies, the second order terms are the same as those in Eqs. (22)–(25), but the mean variation of the mean anomaly in Eq. (21) is now simplified to

$$\begin{aligned}F_{1,2} = & \, n\frac{R_\oplus^4}{p^4}\frac{3}{64}\eta\big\{5\eta^2(5s^4+8s^2-8)+16\eta(3s^2-2)^2+15(7s^4-16s^2+8)-2 \\ & \times\big[5\eta^2(15s^2-14)+32\eta(5s^2-4)+8\frac{2\eta+3}{(\eta+1)^2}(5s^2-4)-3(75s^2-62)\big] \\ & \times s^2\cos 2g\big\}. \qquad (28)\end{aligned}$$

As expected, the mean variation of the Delaunay action vanishes also at the third order, which is in full agreement with the Hamiltonian case.

5 An illustrative test case

With the only aim of illustrating the intrinsic features of the essentially different theories discussed above, we discuss the details of an example semi-analytical propagation. To avoid possible troubles due to the singularity of Delaunay variables for circular orbit that might adulterate the analysis, our test are applied to an orbit with moderate eccentricity. More precisely, the initial conditions correspond to the elliptic orbit tested in [50], with $a = 9500$ km, $e = 0.2$, and $i = 20^\circ$, which we integrate semi-analytically with both the non-canonical and canonical solutions for a 3-day interval. The predictions provided by each perturbation solution are then compared with a "true" orbit obtained from the numerical integration of the main problem in Cartesian coordinates. Errors for the coordinates as well as elements are provided for each theory in the following plots. For the latter we computed the errors of the traditional Keplerian elements for greater insight.

5.1 1st order theories

We remind that a first-order theory is made of mean variation equations that include up to second order terms of J_2, to be numerically integrated, a mean-to-osculating transformation that is restricted to first order effects, and a first-order osculating-to-mean transformation that is improved with the inverse second order terms of the Delaunay action for initialization purposes.

After computing the semi-analytical solution in Delaunay variables, they are transformed into Cartesian coordinates, from which we compute the Root Square sum of the errors with respect to the true orbit. As shown in Fig. 1, the errors of the coordinates are dominated by the inaccuracies resulting from the truncation of the mean-to-osculating transformation to first order effects, which for this short propagation interval hidden the contribution of the errors stemming from the second-order truncation of the mean variations. Rather than extending the propagation interval, later we will refine the semi-analytical solutions to clearly show this effect. While the non-canonical solution seems slightly more accurate for this particular example, both semi-analytical solution remain comparable and remain under the expected accuracy of a properly calibrated first order solution.

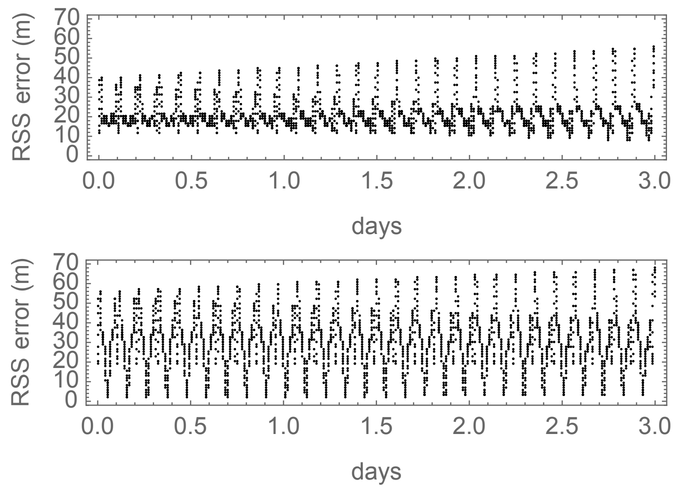

Figure 1: Position errors of the test case. Top: non-canonical theory; Bottom: canonical theory.

Figure 2 presents the errors of the osculating elements superimposed to a linear fit to them. The only relevant differences are observed for the semimajor axis, where the errors stemming from the non-canonical theory average to just a few cm whereas they notably shift to about 5 m, about two orders of magnitude higher, in the case of the canonical theory, and for the eccentricity, where the shift of the average is about one order of magnitude higher for the canonical solution. Like for the coordinates, the amplitude of the periodic errors dominates the picture with the exception of the right ascension of the ascending node, where a secular trend of a few hundredths of arc second per day is clearly apparent in both cases. Similar trends in ω and M remain hidden by the notably higher amplitude of the periodic components of the errors. So the main feature of the non-canonical theory of providing a better agreement with the orbital elements' true dynamics is demonstrated.

5.2 Refinements of the mean-to-osculating transformation

To get a better insight on the differences between the non-canonical and canonical approaches, we supplement both semi-analytical solutions with the second-order terms

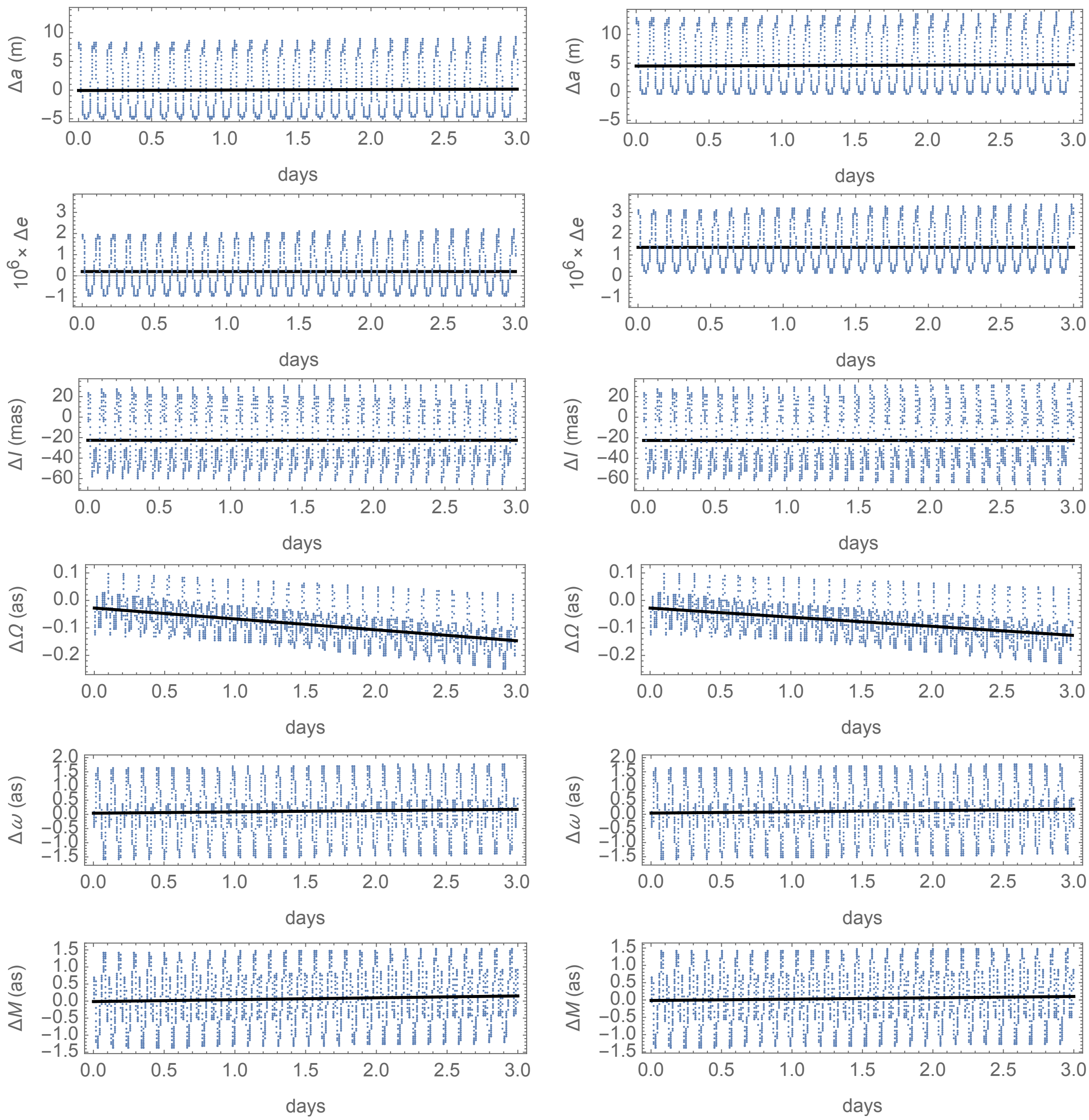

Figure 2: Orbital elements errors of the test case. Left: non-canonical theory; Right: canonical theory.

of the respective mean to osculating transformation. As shown in Fig. 3, now the errors of the coordinates clearly disclose their respective secular trends, in this way supporting the theoretical features discussed in the development of each theory, which completely agree with the numerical simulation. Namely, due to the existing terms in the mean variation of the Delaunay action that are neglected by the truncation, the secular errors grow at a slightly higher rate in the non-canonical case. Still, the periodic corrections are better captured by the non-canonical mean to osculating transformation.

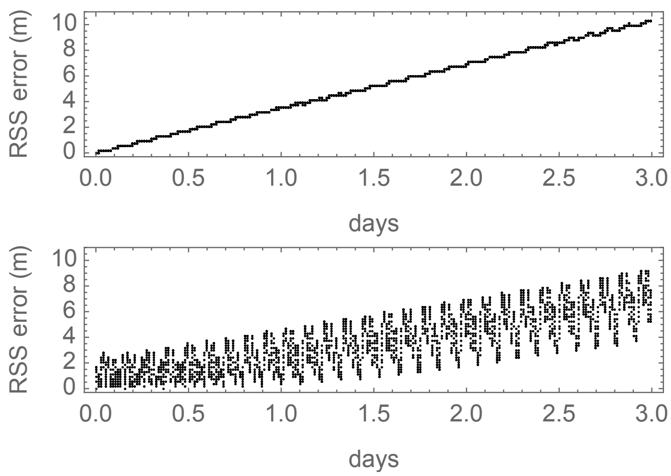

Figure 3: Position errors of the improved theories. Top: non-canonical case; Bottom: canonical case.

Regarding the errors of the osculating elements, while the basic features shown by the first-order theory remain, the shifts from the zero average of the errors of the semimajor axis and the eccentricity are now of the same order of magnitude, as clearly observed in Fig. 4. In general, the canonical solution shows a slightly worse performance than the non-canonical one in what respects to the amplitude of the periodic errors. That is, the improvements in the mean-to-osculating transformation bring now both theories to an analogous level of accuracy. On the other hand, the effect of the truncation of the mean variation of the Delaunay action to second order terms, which has no effects in the canonical case, is now clearly apparent in the time history of the errors of the mean anomaly obtained with the non-canonical solution.

Conclusions

The second-order terms of the pure periodic, non-canonical transformation yielding the exact separation of long- and short-period effects have been computed for the first time for the main problem of artificial satellite theory. The non-canonical transformation was computed for the Delaunay canonical variables, with the consequent drawbacks in the semi-analytical propagation of the lower eccentricity orbits. Recomputation of the solution in non-singular variables, either canonical or not, would follow analogous steps to the ones described here. On the other hand, the use of Delaunay canonical

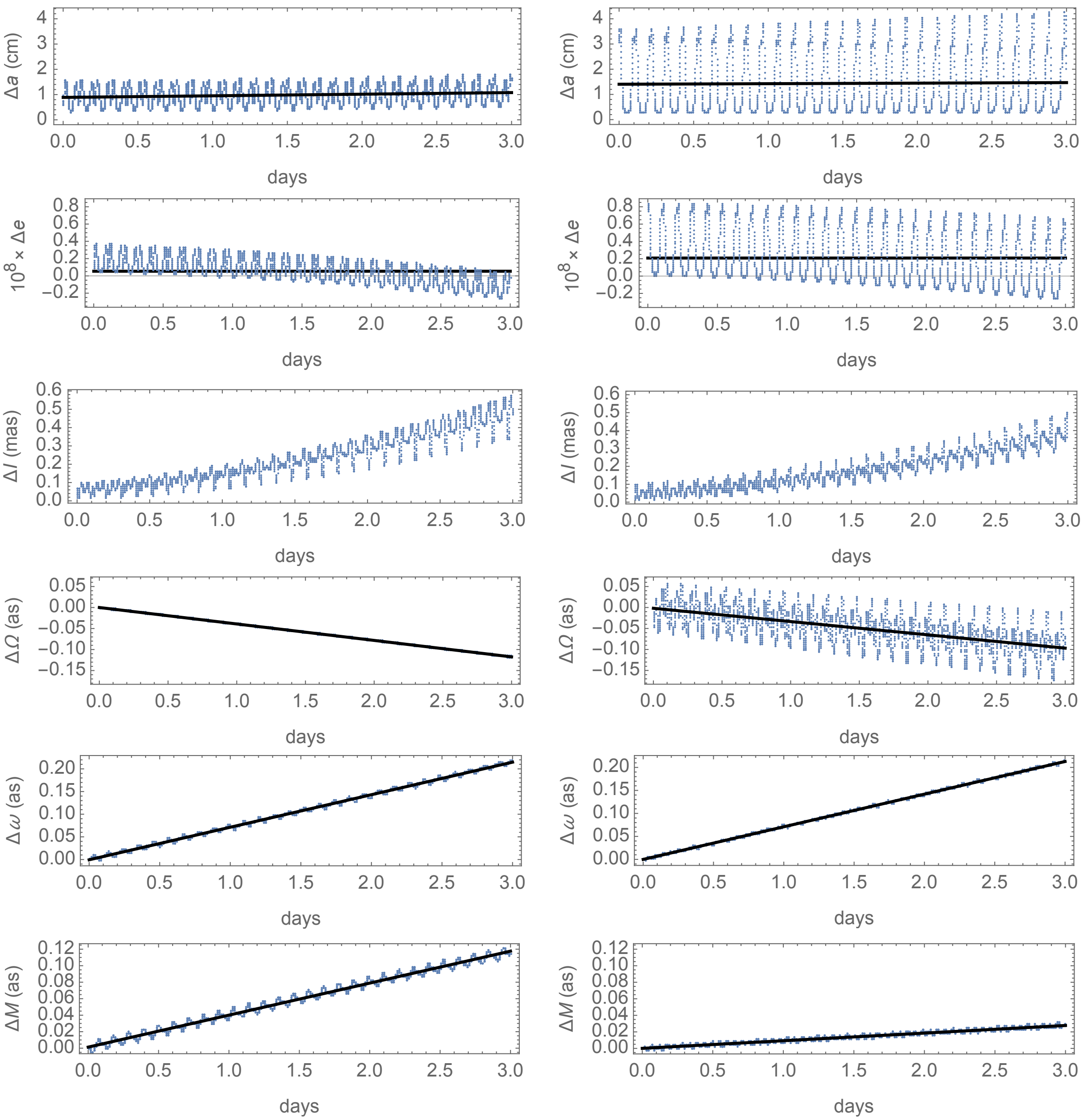

Figure 4: Orbital elements errors of the improved non-canonical (left) and canonical theories (right).

variables served us to check the canonical character of a mean to osculating transformation based on the choice of a generator that is purely periodic in the mean anomaly. Results for the main problem confirm previous conjectures based on simplified models. To wit, the non-canonical solution results in a closer approach to the mean elements dynamics due to a better handling of the short-period components of it. However, this appealing advantage may be counterbalanced by the fact that the mean variation of the semimajor axis, or of other analogous element, no longer vanishes beyond the second order of the theory. In consequence, due to the unavoidable truncation of perturbation solutions, the neglected secular and long-period terms of this mean variation may have a negative impact in the integration of the mean motion. Therefore, this kind of non-canonical solution may suffer from faster deterioration of the errors in the propagation of coordinates.

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A Integration of functions of the equation of the center

In spite of the mean anomaly remains implicit in the equations through the explicit appearance of the true anomaly $f = f(\ell, e)$, most of the integrals to be solved in the perturbation approach can be reduced to cases already reported in the literature [51, 52, 53, 19, 20, 21]. The remaining integrals, which involve functions related to the equation of the center, can be approached by integration by parts, in which process we find again integrals of known types [22].

In particular, as soon as in the computation of second order terms, we find integrals of the form [54]

$$\begin{aligned} \mathcal{C}_m &= \int \phi^m \frac{p^2}{r^2} \cos(mf + \xi)\, \mathrm{d}\ell, \\ \mathcal{S}_m &= \int \phi^m \frac{p^2}{r^2} \sin(mf + \xi)\, \mathrm{d}\ell, \end{aligned}$$

where $\mathrm{d}\xi/\mathrm{d}\ell = 0$. Noting that $\mathcal{C}_m = (\eta^3/m) \int \phi^m \mathrm{d}[\sin(mf + \xi)]$, and using integration by parts, for $m = 2$ we obtain

$$\begin{aligned} \frac{m}{\eta^3}\mathcal{C}_2 = \phi^2 \sin(mf + \xi) - \frac{2}{\eta^3}\mathcal{S}_1 \\ + 2 \int \phi \sin(mf + \xi)\, \mathrm{d}\ell, \end{aligned} \tag{29}$$

where $\mathcal{S}_1 = \int (p/r)^2 \phi \sin(mf + \xi)\, \mathrm{d}\ell$ is in turn integrated by parts, to give

$$\begin{aligned} \frac{m}{\eta^3}\mathcal{S}_1 = & -\phi \cos(mf + \xi) + \frac{1}{m} \sin(mf + \xi) \\ & - \int \cos(mf + \xi)\, \mathrm{d}\ell, \end{aligned} \tag{30}$$

a result that was first reported in [22]. Hence, Eq. (29) turns into

$$\frac{m}{\eta^3}\mathcal{C}_2 = \phi^2 \sin(mf + \xi) + \frac{2}{m} \phi \cos(mf + \xi)$$

$$
-\frac{2}{m^2}\sin(mf+\xi)+\frac{2}{m}\int\cos(mf+\xi)\,\mathrm{d}\ell
+2\int\phi\sin(mf+\xi)\,\mathrm{d}\ell. \tag{31}
$$

Rules for the integration of $\cos(mf + x)$ in the mean anomaly are detailed in [19], whereas $\int\phi\sin(mf+\xi)\,\mathrm{d}\ell$ corresponds to case (v) of [21].

Analogously, from the previous result and on account of the fundamental theorem of calculus, we readily obtain

$$
\frac{1}{2\pi}\int_0^{2\pi}\phi^2\frac{p^2}{r^2}\cos(mf+\xi)\,\mathrm{d}\ell =
\frac{\eta^3}{m}\frac{2}{m}\frac{1}{2\pi}\int_0^{2\pi}\cos(mf+\xi)\,\mathrm{d}\ell
+2\frac{\eta^3}{m}\frac{1}{2\pi}\int_0^{2\pi}\phi\sin(mf+\xi)\,\mathrm{d}\ell, \tag{32}
$$

where both quadratures are readily integrated following the rules in [52] for the first, and in [20] for the second.