Minimum Non-Chromatic-λ-Choosable Graphs

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Minimum non-chromatic-λ-choosable graphs

(Extended abstract)

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Abstract

For a multi-set $\lambda = \{k_1, k_2, \ldots, k_q\}$ of positive integers, let $k_\lambda = \sum_{i=1}^{q} k_i$. A λ-list assignment of G is a list assignment L of G such that the colour set $\bigcup_{v \in V(G)} L(v)$ can be partitioned into the disjoint union $C_1 \cup C_2 \cup \ldots \cup C_q$ of q sets so that for each i and each vertex v of G, $|L(v) \cap C_i| \geq k_i$. We say G is λ-choosable if G is L-colourable for any λ-list assignment L of G. The concept of λ-choosability puts k-colourability and k-choosability in the same framework: If $\lambda = \{k\}$, then λ-choosability is equivalent to k-choosability; if λ consists of k copies of 1, then λ-choosability is equivalent to k-colourability. If G is λ-choosable, then G is k_λ-colourable. On the other hand, there are k_λ-colourable graphs that are not λ-choosable, provided that λ contains an integer larger than 1. Let $\phi(\lambda)$ be the minimum number of vertices in a k_λ-colourable non-λ-choosable graph. This paper determines the value of $\phi(\lambda)$ for all λ.

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1 Introduction

A *proper colouring* of a graph G is a mapping $f : V(G) \to \mathbb{N}$ such that $f(u) \neq f(v)$ for any edge uv of $E(G)$. The *chromatic number* $\chi(G)$ of G is the minimum positive integer k such that G is k-colourable, i.e., there is a proper colouring f of G using colours from $\{1, 2, \ldots, k\}$. The *choice number* $ch(G)$ of G is the minimum positive integer k such that G is k-choosable, i.e., if L is a list assignment which assigns to each vertex v a set $L(v) \subseteq \mathbb{N}$

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of at least k integers as permissible colours, then there is a proper colouring f of G such that $f(v) \in L(v)$ for each vertex v.

It follows from the definitions that $\chi(G) \leq ch(G)$ for any graph G, and it was shown in [5] that bipartite graphs can have arbitrarily large choice number. An interesting problem is for which graphs G, $\chi(G) = ch(G)$. Such graphs are called *chromatic-choosable*. Chromatic-choosable graphs have been studied extensively in the literature. There are a few challenging conjectures that assert certain families of graphs are chromatic-choosable. The most famous problem concerning this concept is perhaps the *list colouring conjecture*, which asserts that line graphs are chromatic-choosable [1]. Another problem concerning chromatic-choosable graphs that has attracted a lot of attention is the minimum order of a non-chromatic-choosable graph with given chromatic number. For a positive integer k, let

$$\phi(k) = \min\{n : \text{ there exists a non-}k\text{-choosable } k\text{-chromatic } n\text{-vertex graph}\}.$$

Ohba [20] conjectured that $\phi(k) \geq 2k+2$. In other words, k-colourable graphs on at most $2k+1$ vertices are k-choosable. This conjecture was studied in many papers [14, 16, 18–22, 24, 25], and was finally confirmed by Noel, Reed and Wu [18]. This lower bound is tight if k is even, i.e., $\phi(k) = 2k+2$ when k is even. Noel [17] further conjectured that if k is odd, then k-colourable graphs on at most $2k+2$ vertices are also k-choosable. Recently, the authors of this paper confirmed Noel's conjecture [28], and determined the value of $\phi(k)$ for all k.

Theorem 1. *[28] For $k \geq 2$,*

$$\phi(k) = \begin{cases} 2k+2, & \text{if } k \text{ is even,} \\ 2k+3, & \text{if } k \text{ is odd.} \end{cases}$$

The concept of λ-choosability is a refinement of choosability introduced in [32]. Assume that $\lambda = \{k_1, k_2, \ldots, k_q\}$ is a multi-set of positive integers. Let $k_\lambda = \sum_{i=1}^q k_i$ and $|\lambda| = q$. A *λ-list assignment* of G is a list assignment L such that the colour set $\bigcup_{v \in V(G)} L(v)$ can be partitioned into the disjoint union $C_1 \cup C_2 \cup \ldots \cup C_q$ of q sets so that for each i and each vertex v of G, $|L(v) \cap C_i| \geq k_i$. Note that for each vertex v, $|L(v)| \geq \sum_{i=1}^q k_i = k_\lambda$. So a λ-list assignment L is a k_λ-list assignment with some restrictions on the set of possible lists. We say G is *λ-choosable* if G is L-colourable for any λ-list assignment L of G.

For a positive integer a, let $m_\lambda(a)$ be the multiplicity of a in λ. If $m_\lambda(a) = m$, then instead of writing m times the integer a, we may write $a \star m$. For example, $\lambda = \{1 \star k_1, 2 \star k_2, 3\}$ means that λ is a multi-set consisting of k_1 copies of 1, k_2 copies of 2 and one copy of 3. If $\lambda = \{k\}$, then λ-choosability is the same as k-choosability; if $\lambda = \{1 \star k\}$, then λ-choosability is equivalent to k-colourability [32]. So the concept of λ-choosability puts k-choosability and k-colourability in the same framework.

Assume that $\lambda = \{k_1, k_2, \ldots, k_q\}$ and $\lambda' = \{k'_1, k'_2, \ldots, k'_p\}$. We say λ' is a *refinement* of λ if $p \geq q$ and there is a partition $I_1 \cup I_2 \cup \ldots \cup I_q$ of $\{1, 2, \ldots, p\}$ such that $\sum_{j \in I_t} k'_j = k_t$ for $t = 1, 2, \ldots, q$. We say λ' is obtained from λ by increasing some parts if $p = q$ and

$k_t \le k'_t$ for $t = 1, 2, \dots, q$. We write $\lambda \le \lambda'$ if λ' is a refinement of λ'', and λ'' is obtained from λ by increasing some parts. It follows from the definitions that if $\lambda \le \lambda'$, then every λ-choosable graph is λ'-choosable. Conversely, it was proved in [32] that if $\lambda \nleq \lambda'$, then there is a λ-choosable graph which is not λ'-choosable. In particular, λ-choosability implies k_λ-colourability, and if $\lambda \ne \{1 \star k_\lambda\}$, then there are k_λ-colourable graphs that are not λ-choosable.

All the partitions λ of a positive integer k are sandwiched between $\{k\}$ and $\{1 \star k\}$ in the above order. As observed above, $\{k\}$-choosability is the same as k-choosability, and $\{1 \star k\}$-choosability is equivalent to k-colourability. For other partitions λ of k, λ-choosability reveals a complex hierarchy of colourability of graphs sandwiched between k-colourability and k-choosability. The framework of λ-choosability provides room to explore generalizations of colourability and choosability results or problems (see [8, 10, 32])

2 Preliminaries

In this paper, we are interested in Ohba type question for λ-choobility. Similar to the definition of $\phi(k)$, for a multi-set λ of positive integers, we define $\phi(\lambda)$ as follows:

Definition 1. *Assume λ is a multi-set of positive integers. Let*

$$\phi(\lambda) = \min\{n : \text{ there exists a non-}\lambda\text{-choosable } k_\lambda\text{-chromatic } n\text{-vertex graph}\}.$$

If $\lambda = \{1 \star k\}$, then λ-choosable is equivalent to k-colourable. In this case, we set $\phi(\lambda) = \infty$. We call such a multi-set λ trivial. In the following, we only consider non-trivial multi-sets of positive integers.

If $\lambda = \{k\}$, then $\phi(\lambda) = \phi(k)$. The value of $\phi(k)$ is determined in Theorem 1. For general multiset λ of positive integers, the function $\phi(\lambda)$ was first studied in [30]. Let $m_\lambda(\text{odd})$ be the number of odd integers in λ. The following result was proved in [30].

Theorem 2. *For any non-trivial multi-set λ of positive integers,*

$$2k_\lambda + m_\lambda(1) + 2 \leqslant \phi(\lambda) \leqslant \min\{2k_\lambda + m_\lambda(\text{odd}) + 2, 2k_\lambda + 5m_\lambda(1) + 3\}.$$

If $m_\lambda(1) = m_\lambda(\text{odd}) = t$, then it follows from Theorem 2 that $\phi(\lambda) = 2k_\lambda + t + 2$. However, when $m_\lambda(1)$ and $m_\lambda(\text{odd}) - m_\lambda(1)$ are both large, then the gap between the upper and lower bounds for $\phi(\lambda)$ in Theorem 2 becomes large.

3 Main result

This paper proves Theorem 3 below, which strengthens Theorem 1 and Theorem 2 and determines the value of $\phi(\lambda)$ for all λ.

Theorem 3. *Assume λ is a non-trivial multi-set of positive integers. Then*

$$\phi(\lambda) = \min\{2k_\lambda + m_\lambda(\text{odd}) + 2, 2k_\lambda + 3m_\lambda(1) + 3\}.$$

Below is a sketch of the proof of Theorem 3.

By Theorem 2, to prove Theorem 3, it suffices to consider the case that $m_\lambda(\text{odd}) > m_\lambda(1)$.

First we consider the case that $m_\lambda(1) = 0$ and $m_\lambda(\text{odd}) > 0$. In this case, we need to show that $\phi(\lambda) = 2k_\lambda + 3$.

Let $k_\lambda = k$. By Theorem 2, $2k+2 \le \phi(\lambda) \le 2k+3$. So it suffices to show that $\phi(\lambda) \ne 2k+2$, i.e., any graph G with $\chi(G) \le k$ and $|V(G)| \le 2k+2$ is λ-choosable. We only need to consider the case that G is a complete k-partite graph. The following result was proved in [29].

Theorem 4. *Assume G is a complete k-partite graph with $|V(G)| \le 2k+2$. Then G is k-choosable, unless k is even and $G = K_{4,2\star(k-1)}$ or $G = K_{3\star(k/2+1),1\star(k/2-1)}$.*

Thus we may assume that k is even and $G = K_{4,2\star(k-1)}$ or $G = K_{3\star(k/2+1),1\star(k/2-1)}$. We say a k-list assignment L of G is *bad* if G is not L-colourable. All bad assignments for $K_{4,2\star(k-1)}$ and $K_{3\star(k/2+1),1\star(k/2-1)}$ are characterized in [4] and [29], respectively and we can verify that such bad list assignments is not λ-list assignment (using the assumption $m_\lambda(\text{odd}) > 0$). This implies that all graphs $K_{4,2\star(k-1)}$ and $K_{3\star(k/2+1),1\star(k/2-1)}$ are λ-choosable. This completes the proof for the case $m_\lambda(1) = 0$.

Next we consider the case that $m_\lambda(1) = a \ge 1$ and $m_\lambda(\text{odd}) - m_\lambda(1) = c \ge 1$. We need to show that $\phi(\lambda) = \min\{2k+a+c+2, 2k+3a+3\}$. First, we prove the upper bound, i.e.,

$$\phi(\lambda) \le \min\{2k+a+c+2, 2k+3a+3\}.$$

By Theorem 2, $\phi(\lambda) \le 2k+a+c+2$. It remains to show that $\phi(\lambda) \le 2k+3a+3$. Observe that $k_\lambda = k$, $m_\lambda(1) = a$ and $m_\lambda(\text{odd}) = a+c$ implies that $\{1\star a, 2\star(k-a-3c)/2, 3\star c\}$ is a refinement of λ. Hence it suffices to prove the following lemma.

Lemma 5. *Assume $\lambda = \{1\star a, 2\star b, 3\star c\}$ and $k = a+2b+3c$ (and hence $m_\lambda(1) = a$, $m_\lambda(\text{odd}) = a+c$ and $k_\lambda = k$). Then there exists a k-chromatic graph G with $|V(G)| = 2k+3a+3$ which is not λ-choosable.*

Let $G = K_{5\star(a+1),2\star(k-a-1)}$ be the complete k-partite graph with partite sets $U_i = \{u_{i,1}, u_{i,2}, u_{i,3}, u_{i,4}, u_{i,5}\}$ where $i = 1,2,\dots,a+1$, and $V_j = \{v_{j,1}, v_{j,2}\}$ where $j = 1,2,\dots,k-a-1$.

Let $S_i = \{s_{i,1}, s_{i,2}, \dots, s_{i,6}\}$ be pairwise disjoint sets of size 6 where $i = 1,2,\dots,c$ and let $T_i = \{t_{i,1}, t_{i,2}, t_{i,3}, t_{i,4}\}$ be pairwise disjoint sets of size 4 where $i = 1,2,\dots,b$. Let E be a set of a colours, and the sets E, S_i, T_i are pairwise disjoint and let

$$A_1 = \bigcup_{i=1}^{c}\{s_{i,1}, s_{i,3}, s_{i,5}\},\ A_2 = \bigcup_{i=1}^{c}\{s_{i,1}, s_{i,3}, s_{i,6}\},\ A_3 = \bigcup_{i=1}^{c}\{s_{i,1}, s_{i,2}, s_{i,4}\},\ A_4 = \bigcup_{i=1}^{c}\{s_{i,2}, s_{i,3}, s_{i,4}\},$$
$$A_5 = \bigcup_{i=1}^{c}\{s_{i,2}, s_{i,5}, s_{i,6}\},\ A_6 = \bigcup_{i=1}^{c}\{s_{i,1}, s_{i,2}, s_{i,3}\},\ A_7 = \bigcup_{i=1}^{c}\{s_{i,4}, s_{i,5}, s_{i,6}\},$$
$$B_1 = \bigcup_{i=1}^{b}\{t_{i,2}, t_{i,3}\},\ B_2 = \bigcup_{i=1}^{b}\{t_{i,2}, t_{i,4}\},\ B_3 = \bigcup_{i=1}^{b}\{t_{i,1}, t_{i,2}\},\ B_4 = \bigcup_{i=1}^{b}\{t_{i,1}, t_{i,3}\},$$
$$B_5 = \bigcup_{i=1}^{b}\{t_{i,1}, t_{i,4}\},\ B_6 = \bigcup_{i=1}^{b}\{t_{i,1}, t_{i,2}\},\ B_7 = \bigcup_{i=1}^{b}\{t_{i,3}, t_{i,4}\}.$$

Let L be the λ-list assignment of G defined as follows:

$$L(v) = \begin{cases} A_j \cup B_j \cup E, & \text{if } v = u_{i,j}, 1 \le i \le a+1, 1 \le j \le 5, \\ A_{j+5} \cup B_{j+5} \cup E, & \text{if } v = v_{i,j}, 1 \le i \le k-a-1, 1 \le j \le 2, \end{cases}$$

It can be proved that L is λ-list assignment and G is not L-colourable. The proof is a little complicated, and the details are omitted.

It remains to prove the lower bound that $\phi(\lambda) \geqslant \min\{2k+3a+3, 2k+a+c+2\}$.

Assume to the contrary that $\phi(\lambda) < \min\{2k+a+c+2, 2k+3a+3\}$ for some λ. We choose such a multi-set $\lambda = \{k_1, k_2, \ldots, k_q\}$ with $|\lambda| = q$ minimum. Assume that $k_1 = k_2 = \ldots = k_a = 1$ and $3 \le k_{a+1} \le k_{a+2} \le \ldots \le k_{a+c}$ are the odd integers in λ.

Let $n = \min\{2k+a+c+2, 2k+3a+3\}$. Then there is a k-chromatic graph G with $|V(G)| \le n-1$ which is not λ-choosable. We may assume that G is a complete k-partite graph with $|V(G)| = n-1$ and with partite sets $P_1, P_2, \ldots, P_k$ such that $|P_1| \ge |P_2| \ge \ldots \ge |P_k|$. For a positive integer i, let

$$I_i = \{j : |P_j| = i\}.$$

Note that $|P_1| \ge 3$ (as $|V(G)| > 2k$). Using the assumption $m_\lambda(1) \ge 1$ and the minimality of $|\lambda|$, we can conclude that $|P_1| \le 4$, and if $c \le 2a+1$, then $|P_1| \le c-2a+3$. Since $a \ge 1$, we know that $c \ge 2a \ge 2$, and if $c = 2$, then $a = 1$ and $|P_1| = 3$.

Definition 2. *A 4-tuple (a_1, a_2, a_3, a_4) of integers is* reducible *if*

$$0 \le a_i \le |I_i|, \ \sum_{i=1}^{4} a_i = k_{a+1} \ \textit{and} \ 2k_{a+1}+1 \le \sum_{i=1}^{4} i a_i \le 2k_{a+1}+2.$$

Combining with Theorem 4 and the minimality of $|\lambda|$, we conclude that

Claim 6. *There is no reducible 4-tuple.*

It follows from Claim 6 that $|I_2| \le k_{a+1}-2$ and if $c \ge 3$, then $|I_1| \ge \frac{2}{3}k_{a+1}$ and if $c = 2$, then $|I_1| \ge (k_{a+1}-1)/2$. Recall that $3 \le |P_1| \le 4$. By Claim 6, we can conclude that if $|P_1| = 4$, then $|I_3| < \lfloor \frac{k_{a+1}-|I_2|-1}{2} \rfloor$, $|I_4| < \lceil \frac{k_{a+1}-|I_2|-2|I_3|-1}{3} \rceil + 1$ and if $|P_1| = 3$, then $|I_3| < \lceil \frac{k_{a+1}-|I_2|-1}{2} \rceil + 1$. This contradicts to $|V(G)| = n-1 \ge 2k+1$. This completes the proof of Theorem 3.

References

[1] Béla Bollobás and Andrew Harris. List-colourings of graphs. *Graphs Combin.*, 1(2):115–127, 1985.

[2] Hojin Choi and Young Soo Kwon. On t-common list-colorings. *Electron. J. Combin.*, 24(3):Paper 3.32, 10, 2017.

[3] Lech Duraj, Grzegorz Gutowski, and Jakub Kozik. Chip games and paintability. *Electron. J. Combin.*, 23(3):Paper 3.3, 12, 2016.

[4] Hikoe Enomoto, Kyoji Ohba, Katsuhiro Ota, and Junko Sakamoto. Choice number of some complete multi-partite graphs. *Discrete Math.*, 244(1-3):55–66, 2002.

[5] Paul Erdős, Arthur L. Rubin, and Herbert Taylor. Choosability in graphs. In *Proceedings of the West Coast Conference on Combinatorics, Graph Theory and Computing (Humboldt State Univ., Arcata, Calif., 1979)*, Congress. Numer., XXVI, pages 125–157. Utilitas Math., Winnipeg, Man., 1980.

[6] Fred Galvin. The list chromatic index of a bipartite multigraph. *J. Combin. Theory Ser. B*, 63(1):153–158, 1995.

[7] Sylvain Gravier and $Frdric$ Maffray. Choice number of 3-colorable elementary graphs. *Discrete Math.*, 165/166:353–358, 1997. Graphs and combinatorics (Marseille, 1995).

[8] Yangyan Gu, Yiting Jiang, David Wood, and Xuding Zhu. Refined list version of Hadwiger's conjecture. arXiv:2209.07013.

[9] Po-Yi Huang, Tsai-Lien Wong, and Xuding Zhu. Application of polynomial method to on-line list colouring of graphs. *European J. Combin.*, 33(5):872–883, 2012.

[10] Arnfried Kemnitz and Margit Voigt. A note on non-4-list colorable planar graphs. *Electron. J. Combin.*, 25(2):Paper 2.46, 5, 2018.

[11] Seog-Jin Kim, Young Soo Kwon, Daphne Der-Fen Liu, and Xuding Zhu. On-line list colouring of complete multipartite graphs. *Electron. J. Combin.*, 19(1):Paper 41, 13, 2012.

[12] Seog-Jin Kim and Boram Park. Bipartite graphs whose squares are not chromatic-choosable. *Electron. J. Combin.*, 22(1):Paper 1.46, 12, 2015.

[13] Seog-Jin Kim and Boram Park. Counterexamples to the list square coloring conjecture. *J. Graph Theory*, 78(4):239–247, 2015.

[14] Alexandr V. Kostochka, Michael Stiebitz, and Douglas R. Woodall. Ohba's conjecture for graphs with independence number five. *Discrete Math.*, 311(12):996–1005, 2011.

[15] Alexandr V. Kostochka and Douglas R. Woodall. Choosability conjectures and multicircuits. *Discrete Math.*, 240(1-3):123–143, 2001.

[16] Jakub Kozik, Piotr Micek, and Xuding Zhu. Towards an on-line version of $Ohba's$ conjecture. *European J. Combin.*, 36:110–121, 2014.

[17] Jonathan A. Noel. Choosability of graphs with bounded order: Ohba's conjecture and beyond. Master's thesis, McGill University, 2013.

[18] Jonathan A. Noel, Bruce A. Reed, and Hehui Wu. A proof of a conjecture of Ohba. *J. Graph Theory*, 79(2):86–102, 2015.

[19] Jonathan A. Noel, Douglas B. West, Hehui Wu, and Xuding Zhu. Beyond Ohba's conjecture: a bound on the choice number of k-chromatic graphs with n vertices. *European J. Combin.*, 43:295–305, 2015.

[20] Kyoji Ohba. On chromatic-choosable graphs. *J. Graph Theory*, 40(2):130–135, 2002.

[21] Kyoji Ohba. Choice number of complete multipartite graphs with part size at most three. *Ars Combin.*, 72:133–139, 2004.

[22] Bruce Reed and Benny Sudakov. List colouring when the chromatic number is close to the order of the graph. *Combinatorica*, 25(1):117–123, 2005.

[23] Uwe Schauz. Mr. Paint and Mrs. Correct. *Electron. J. Combin.*, 16(1):Research Paper 77, 18, 2009.

[24] Yufa Shen, Wenjie He, Guoping Zheng, and Yanpo Li. Ohba's conjecture is true for graphs with independence number at most three. *Appl. Math. Lett.*, 22(6):938–942, 2009.

[25] Yufa Shen, Wenjie He, Guoping Zheng, Yanning Wang, and Lingmin Zhang. On choosability of some complete multipartite graphs and Ohba's conjecture. *Discrete Math.*, 308(1):136–143, 2008.

[26] M. Voigt. A non-3-choosable planar graph without cycles of length 4 and 5. *Discrete Math.*, 307(7-8):1013–1015, 2007.

[27] Margit Voigt. List colourings of planar graphs. *Discrete Math.*, 120(1-3):215–219, 1993.

[28] Jialu Zhu and Xuding Zhu. Bad list assignments for non-k-choosable k-chromatic graphs with $2k+2$-vertices. arXiv:2202.09756.

[29] Jialu Zhu and Xuding Zhu. Minimum non-chromatic-choosable graphs with given chromatic number. arXiv:2201.02060.

[30] Jialu Zhu and Xuding Zhu. Chromatic λ-choosable and λ-paintable graphs. *J. Graph Theory*, 98(4):642–652, 2021.

[31] Xuding Zhu. On-line list colouring of graphs. *Electron. J. Combin.*, 16(1):Research Paper 127, 16, 2009.

[32] Xuding Zhu. A refinement of choosability of graphs. *J. Combin. Theory Ser. B*, 141:143–164, 2020.