On Three Types of Universal Propositional Proof Systems for All Versions of Many-Valued Logics and Some Its Properties

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Our research refers to the problem of constructing the universal proof systems for all versions of propositional many-valued logics (MVLs) such that any propositional proof system for every variant of MVL can be presented in described form. We presented three types of such universal systems, and investigated some properties of them. The first introduced system **UE** is based on the generalization of the notion of determinative disjunctive normal form [1], the second system **UGS** is based on the generalization of splitting method, described in [2] and the third one **US** is a Gentzen-like system [1].

Let E_k be the set $\left\{0, \frac{1}{k-1}, \ldots, \frac{k-2}{k-1}, 1\right\}$. We use the well-known notions of propositional k-valued formula, defined as usual from propositional variables with values from E_k, (may also be propositional constants) and logical connectives, each of which can be defined by different well known modes. For propositional variable p and $\delta = \frac{i}{k-1}$ $(0 \le i \le k-1)$ we defined additionally "exponent" functions: (1) exponent p as $(p \supset \delta)\ \ \&\ \ (\delta \supset p)$ with Łukasiewicz's implication and (2) exponent p^δ as p with $(k-1)i$ cyclically permuting negation. Then we introduced the additional notion of formula: for every formulas A and B the expression A^B (for both modes) is also formula. In every MVL either only **1** or every of the values $\frac{1}{2} \le \frac{i}{k-1} \le 1$ can be fixed as ***designated values***. A formula φ with variables $p_1, p_2, \ldots, p_n$ is called k-***tautology*** if for every $\tilde{\delta} = (\delta_1, \delta_2, \ldots, \delta_n) \in E_k^n$ assigning δ_j $(1 \le j \le n)$ to each p_j gives the value 1 (or every of the values $\frac{1}{2} \le \frac{i}{k-1} \le 1$) of φ. For every propositional variable p and $\delta \in E_k$ p^δ in sense of both exponent modes is the **literal**. The conjunct K (term) can be represented simply as a set of literals (no conjunct contains a variable with different measures of exponents simultaneously).

Definition 1. Given $\tilde{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_m) \in E_k^m$, the conjunct $K^\sigma = \left\{p_{i_1}^{\sigma_1}, p_{i_2}^{\sigma_2}, \ldots, p_{i_m}^{\sigma_m}\right\}$ is called $\varphi - \frac{i}{k-1}$-determinative $(0 \le i \le k-1)$, if assigning σ_j $(1 \le j \le m)$ to each p_{i_j}, we obtain the value $\frac{i}{k-1}$ of φ independently of the values of the remaining variables.

Definition 2. A DNF $D = \{K_1, K_2, \ldots, K_j\}$ is called determinative DNF (dDNF) for φ if $\varphi = D$ and if "1" (every of the values $\frac{1}{2} \le \frac{i}{k-1} \le 1$) is (are) fixed as designated value, then every conjunct K_i $(1 \le i \le j)$ is 1-determinative ($\frac{i}{k-1}$-determinative) for φ.

Definition of universal elimination system UE [1]. The axioms of Elimination systems **UE** aren't fixed, but for every *k-valued formula* φ each conjunct from some dDNF of φ can be considered as an axiom. For k-valued logic the inference rule is elimination rule (ε-rule)

$$\frac{K_0 \cup \{p^0\}, K_1 \cup \{p^{\frac{1}{k-1}}\}, \ldots, K_{k-2} \cup \{p^{\frac{k-2}{k-1}}\}, K_{k-1} \cup \{p^1\}}{K_0 \cup K_1 \cup \ldots \cup K_{k-2} \cup K_{k-1}}$$

where mutually supplementary literals (variables with corresponding (1) or (2) exponents) are eliminated. Following [1], a finite sequence of conjuncts such that every conjunct in the sequence is one of the axioms of **UE** or is inferred from earlier conjuncts in the sequence by ε-rule is called a proof in **UE**. A DNF $D = \{K_1, K_2, \ldots, K_l\}$ is k-tautological if the empty conjunct $(\emptyset)$ can be proved by using ε-rule from the axioms $\{K_1, K_2, \ldots, K_l\}$.

Definition of universal systems UGS [2]. Let φ be some formula and p be some of its variables. Results of splitting method of formula φ by variable p (*splinted variable*) are the formulas $\varphi[p^\delta]$ for every $\delta \in \left\{0, \frac{1}{k-1}, \ldots, \frac{k-2}{k-1}, 1\right\}$, which are obtained from φ by assigning δ to each occurrence of p. Generalization of splitting method allows to associate with each formula φ some tree with root, nodes of which are labeled by formulas and edges, labeled by literals. The root itself is labeled by formula φ. If some node is labeled by formula v and α is some its variable, then all of k edges, which are going out from this node, are labeled by one of literals α^δ for every δ from the set $\left\{0, \frac{1}{k-1}, \ldots, \frac{k-2}{k-1}, 1\right\}$, and each of k "sons" of this node is labeled by corresponding formula $v[\alpha^\delta]$. Each of the tree's leafs is labeled with some constant from the set $\left\{0, \frac{1}{k-1}, \ldots, \frac{k-2}{k-1}, 1\right\}$. The proof system **UGS** can be defined as follows: for every formula φ some splitting tree must be constructed and if all tree's leafs are labeled by the value 1 (or by some value $\frac{i}{k-1} \geq \frac{1}{2}$), then formula φ is $1-k$-*tautology* ($\leq 1/2 - k$-*tautology*), and therefore we can consider each of pointed constants as the axioms, and if v is formula, which is label of some splitting tree node, and p is its splinted variable, then the following figure $\frac{v[p^0],\, v\left[p^{\frac{1}{k-1}}\right], \ldots, v\left[p^{\frac{k-2}{k-1}}\right],\, v[p^1]}{v}$ can be considered as some inference rule, hence every above described splitting tree can be consider as some proof of the formula φ in the system **UGS**.

Definition of universal systems US [1]. For every literal C and for any set of literals Γ the axiom sxeme of propositional system **US** is $\Gamma, C \to C$. For every formulas A, B. for any set of literals Γ, for each σ_1, σ_2, σ from the set E_k and for $* \in \{\ \&\ , \vee, \supset\}$ the logical rules of US are:

$$\frac{\Gamma \to A^{\sigma_1} \text{ and } \Gamma \to B^{\sigma_2}}{\Gamma \to (A*B)^{\varphi_*(A,B,\sigma_1,\sigma_2)}}, \quad \frac{\Gamma \to A^{\sigma_1} \text{ and } \Gamma \to B^{\sigma_2}}{\Gamma \to (A^B)^{\varphi_{\exp}(A,B,\sigma_1,\sigma_2)}}, \quad \frac{\Gamma \to A^{\sigma}}{\Gamma \to (\neg A)^{\varphi_\neg(A,\sigma)}},$$

$$\frac{\Gamma, p^0 \vdash A,\ \Gamma, p^{\frac{1}{k-1}} \vdash A, \ldots, \Gamma, p^{\frac{k-2}{k-1}} \vdash A,\ \Gamma, p^1 \vdash A}{\Gamma \vdash A},$$

where many-valued functions $\varphi_*(A,B,\sigma_1,\sigma_2)$, $\varphi_{\exp}(A,B,\sigma_1,\sigma_2)$ and $\varphi_\neg(A,\sigma)$, must be defined individually for each version of MVL such, that (a) formulas $A^{\sigma_1} \supset (B^{\sigma_2} \supset (A*B)^{\varphi_*(A,B,\sigma_1,\sigma_2)}$, $A^{\sigma_1} \supset (B^{\sigma_2} \supset (A^B)^{\varphi_{\exp}(A,B,\sigma_1,\sigma_2)}$ and $A^\sigma \supset (\neg A)^{\varphi_\neg(A,\sigma)}$ must be k-tautology in this version and (b) if for some σ_1, σ_2, σ the value of $\sigma_1 * \sigma_2$ ($\sigma_1^{\sigma_2}$, $\neg\sigma$) is one of ***designed values*** in this version of MVL, then $(\sigma_1 * \sigma_2)^{\varphi_*(\sigma_1,\sigma_2,\sigma_1,\sigma_2)} = \sigma_1 * \sigma_2$, $(\sigma_1^{\sigma_2})^{\varphi_{\exp}(\sigma_1,\sigma_2,\sigma_1,\sigma_2)} = \sigma_1^{\sigma_2}$, $(\neg\sigma)^{\varphi_\neg(\sigma,\sigma)} = \neg\sigma$). **Some algorithm for constructing of these formulas is given.**

Theorem. *The systems UE, UGS and US are complete and sound and for every version of MVL some propositional proof system can be presented in every of mentioned types.*

We compare the proof complexities of the same formulas in descreibed systems, compare in each of them the proof complexities of minimal tautologies and results of substitution in them and as well as many other interesting properties.

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References

[1] Anahit Chubaryan and Artur Khamisyan. Two types of universal proof systems for all variants of many-valued logics and some properties of them. https://doi.org/10.1007/s420044-018-0015-4, 2019.

[2] Anahit Chubaryan. Universal system for many-valued logic, based on splitting method, and some of its properties. www.ijisset.org/articles/2019-2/volume-5-issue-5/, 2019.

Appendix

Examples of US presentations for some versions of MVL.

Here we give the US presentations for some systems MVL.

a) For the first of the constructed systems LN_k (Łukasiewicz's negation) with fixed "1" as the designated value, uses conjunction, disjunction, (1) implication, (1) negation and (1) exponent, and as well as constants $\delta = \frac{i}{k-1}$ $(1 \le i \le k-2)$ for using (1) exponent the functions $\varphi_*(A, B, \sigma_1, \sigma_2)$, $\varphi_{\exp}(A, B, \sigma_1, \sigma_2)$, $\varphi_\neg(A, \sigma)$ are defined as follows:

$$\varphi_*(A, B, \sigma_1, \sigma_2) = \sigma_1 * \sigma_2,$$
$$\varphi_{\exp}(A, B, \sigma_1, \sigma_2) = \sigma_1^{\sigma_2},$$
$$\varphi_\neg(A, \sigma) = \neg\sigma.$$

b) For the second systems CN_3 (cyclically permuting negation) with fixed "1" as the designated value, uses conjunction, disjunction,(2) implication, (2) negation and (2) exponent the functions $\varphi_*(A, B, \sigma_1, \sigma_2)$, $\varphi_{\exp}(A, B, \sigma_1, \sigma_2)$, $\varphi_\neg(A, \sigma)$ are defined as follows:

$$\varphi_\supset(A, B, \sigma_1, \sigma_2) = (\sigma_1 \supset \sigma_2)\&(\neg(A \vee \bar{A}) \vee (\bar{\bar{B}} \supset B)) \vee (\neg(A \vee \bar{\bar{A}})\&\neg(B \vee \bar{\bar{B}})),$$
$$\varphi_\vee(A, B, \sigma_1, \sigma_2) = (\sigma_1 \vee \sigma_2) \vee ((A \supset \bar{A})\&\neg(\bar{B} \vee \bar{\bar{B}})) \vee (\neg(\bar{A} \vee \bar{\bar{A}})\&(B \supset \bar{B})),$$
$$\varphi_\&(A, B, \sigma_1, \sigma_2) = (\sigma_1\&\sigma_2) \vee ((A\&\bar{\bar{A}}) \vee (B\&\bar{B})) \vee ((A\&\bar{A}) \vee (B\&\bar{\bar{B}}))$$
$$\varphi_{\exp}(A, B, \sigma_1, \sigma_2) = \sigma_1^{\sigma_2} \vee (\neg(\sigma_1^{\sigma_2})\&\neg(\neg(A^{\sigma_1}\&\bar{B}^{\sigma_2}) \vee \neg\neg(A^{\sigma_1}\&\bar{B}^{\sigma_2}))) \,,$$
$$\varphi_\neg(A, \sigma) = \neg\sigma \,.$$

c) For $LN_{3,2}$ – Łukasiewicz's logic with fixed "1/2" and "1" as the designated value, which uses conjunction, disjunction, (1) implication, (1) negation and (1) exponent, and as well as constants 0, 1/2 and 1 for using (1) exponent the functions $\varphi_*(A, B, \sigma_1, \sigma_2)$, $\varphi_{\exp}(A, B, \sigma_1, \sigma_2)$, $\varphi_-(A, B, \sigma_1, \sigma_2)$ are defined as follows:

$$\varphi_\supset(A, B, \sigma_1, \sigma_2) = \bar{A} \vee B \vee \overline{\sigma_1} \vee \sigma_2 \,,$$
$$\varphi_\vee(A, B, \sigma_1, \sigma_2) = A \vee B \vee \sigma_1 \vee \sigma_2 \,,$$
$$\varphi_\&(A, B, \sigma_1, \sigma_2) = \left(A \vee \sigma_1 \vee \overline{B^{\sigma_2}}\right) \& \left(B \vee \sigma_2 \vee \overline{A^{\sigma_1}}\right) \,,$$
$$\varphi_{\exp}(A, B, \sigma_1, \sigma_2) = A^B \vee \sigma_1^{\sigma_2} \,,$$
$$\varphi_-(A, B, \sigma_1, \sigma_2) = \bar{A} \vee \overline{\sigma_1} \,.$$