A Peculiar Subset of The Smallest Inductive Set

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December 8, 2019

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Abstract. By ZF, there exists a peculiar subset $\bar{\omega}$ of the smallest inductive set ω, which is not just infinite, but D-finite. If ZF is consistent, then ZF + $\bar{\omega} \neq \omega$ is also consistent; otherwise, ZF is inconsistent. Moreover, if ZF is consistent, then $\bar{\omega}$ and ω are indistinguishable in the forcing method, and so the forcing method has limitation. In order to avoid conflicts with ZF, it will be necessary to discriminate $\bar{\omega}$ from ω in the forcing method, and thus some problems relevant to $\bar{\omega}$ deserve deeper discussion.

1. Introduction

Let ω be the minimal transfinite number. By the Axiom of Infinity, we have

$$\forall x \in \omega \, (x = 0 \vee \exists y \in \omega (x = y^+)) \tag{1.1}$$

where $y^+ = y \cup \{y\}$. Let $F(\mathcal{X}, \mathcal{E})$:

$$\mathrm{Fun}(\mathcal{X}) \wedge \mathrm{Fun}(\mathcal{E}) \wedge \mathrm{dom}(\mathcal{E}) = \Re \wedge \mathrm{ran}(\mathcal{E}) = \mathrm{dom}(\mathcal{X}) = \mathcal{N} \wedge \mathrm{ran}(\mathcal{X}) \subset \omega \tag{1.2}$$

where a predicate $\mathrm{Fun}(\mathcal{X})$ means that $\mathcal{X}$ is a function, $\Re$ is the set of real numbers and $\mathcal{N} = \omega - \{0\}$. Then there are some $\mathcal{X}$ and $\mathcal{E}$ such that

$$\forall E > 0 \, \exists N = \mathcal{E}(E) \, \forall n > N (\mathcal{X}(n) > E). \tag{1.3}$$

Let $A(\mathcal{X}, \mathcal{E})$ denote (1.3). We have

$$\exists \mathcal{E}(F(\mathcal{X}, \mathcal{E}) \wedge A(\mathcal{X}, \mathcal{E})) \Rightarrow \mathcal{X} \to \infty$$

for some $\mathcal{X}$. Now we define

$$\bar{\omega} = \{x \in \omega : \forall \mathcal{X} \, \forall \mathcal{E}(F(\mathcal{X}, \mathcal{E}) \wedge A(\mathcal{X}, \mathcal{E}) \Rightarrow B(x, \mathcal{X}))\} \tag{1.4}$$

where $B(x, \mathcal{X}) : \exists N \, \forall n > N \, (x \in \mathcal{X}(n))$. By the Separation Axioms, $\bar{\omega}$ is exactly a set. Obviously, $0 \in \bar{\omega}$ and $x \in \bar{\omega} \Rightarrow x \in \omega$ holds for every x, i. e.,

$$\bar{\omega} \neq \varnothing \wedge \bar{\omega} \subset \omega. \tag{1.5}$$

So, can we claim that $\bar{\omega} = \omega$?

Date: Jun. 24, 2019.
2000 *Mathematics Subject Classification.* 03E30, 26A03, 03B10, 03E35, 03E10, 03E40, 03E50.
Key words and phrases. ZF; ω; $\bar{\omega}$; infinite; D-finite; consistency; forcing method; CH.
Thanks to all the people who helped me with this paper.

2. ω & $\bar{\omega}$

Lemma 2.1. *Let $\mathcal{Y}$ be the variable function such that $\mathcal{Y} \to \infty$ where*

$$\mathrm{dom}(\mathcal{Y}) = \mathcal{N} \text{ and } \mathrm{ran}(\mathcal{Y}) \subset \omega,$$

and let m be the variable object such that $m \in \mathrm{dom}(\mathcal{Y})$. Then

$$\neg\,(\,\exists \mathcal{Y}\, \forall m\, (\mathcal{Y}(m) \in \bar{\omega})). \tag{2.1}$$

Proof. Let $R(x)$: $\forall \mathcal{X}\, \forall \mathcal{E}\, (F(\mathcal{X}, \mathcal{E}) \wedge A(\mathcal{X}, \mathcal{E}) \Rightarrow B(x, \mathcal{X}))$. By the following properties for the universal and existential quantifiers[1]:

$$\begin{aligned}
(2.2)\quad &(1)\ \forall x\, (Q(x) \Rightarrow S(x)) \equiv \exists x\, Q(x) \Rightarrow \forall x\, S(x),\\
&(2)\ \exists x\, (Q(x) \Rightarrow S(x)) \equiv \forall x\, Q(x) \Rightarrow \exists x\, S(x),\\
&(3)\ \forall x\, (Q(x) \wedge S(x)) \equiv \forall x\, Q(x) \wedge \forall x\, S(x),\\
&(4)\ \exists x\, (Q(x) \wedge S(x)) \Rightarrow \exists x\, Q(x) \wedge \exists x\, S(x) \text{ is a tautology,}
\end{aligned}$$

from (1.4), we have

$$\begin{aligned}
(2.3)\quad & && \forall x\, (x \in \bar{\omega} \Leftrightarrow x \in \omega \wedge R(x))\\
(2.4)\quad & \Leftrightarrow && \forall x\, ((x \in \bar{\omega} \Rightarrow x \in \omega \wedge R(x)) \wedge (x \in \omega \wedge R(x) \Rightarrow x \in \bar{\omega}))\\
(2.5)\quad & \Leftrightarrow && \forall x\, (x \in \bar{\omega} \Rightarrow x \in \omega \wedge R(x)) \wedge \forall x\, (x \in \omega \wedge R(x) \Rightarrow x \in \bar{\omega})
\end{aligned}$$

where

$$\begin{aligned}
(2.6)\quad & && \forall x\, (x \in \bar{\omega} \Rightarrow x \in \omega \wedge R(x))\\
(2.7)\quad & \Leftrightarrow && \exists x (x \in \bar{\omega}) \Rightarrow \forall x (x \in \omega \wedge R(x)) \ (\textit{by (1) of (2.2)})\\
(2.8)\quad & \Leftrightarrow && \exists x (x \in \bar{\omega}) \Rightarrow \forall x (x \in \omega) \wedge \forall x R(x) \ (\textit{by (3) of (2.2)})\\
(2.9)\quad & \Leftrightarrow && \exists x (x \in \bar{\omega}) \Rightarrow \forall x (x \in \omega) \wedge \forall x\, \forall \mathcal{X}\, \forall \mathcal{E} (F(\mathcal{X}, \mathcal{E}) \wedge A(\mathcal{X}, \mathcal{E}) \Rightarrow B(x, \mathcal{X}));
\end{aligned}$$

herein, using the Separation Axioms, and the fact that $\bar{\omega} \neq \varnothing$, we have

$$\begin{aligned}
(2.10)\quad & && \forall x\, \forall \mathcal{X}\, \forall \mathcal{E}\, (\, F(\mathcal{X}, \mathcal{E}) \wedge A(\mathcal{X}, \mathcal{E}) \Rightarrow B(x, \mathcal{X}))\\
(2.11)\quad & \Leftrightarrow && \forall x\, \forall \mathcal{X}\, (\forall \mathcal{E}\, (\, F(\mathcal{X}, \mathcal{E}) \wedge A(\mathcal{X}, \mathcal{E}) \Rightarrow B(x, \mathcal{X})))\\
(2.12)\quad & \Leftrightarrow && \forall x\, \forall \mathcal{X}\, (\, \exists \mathcal{E}\, (F(\mathcal{X}, \mathcal{E}) \wedge A(\mathcal{X}, \mathcal{E})) \Rightarrow \forall \mathcal{E}\, B(x, \mathcal{X}))\\
(2.13)\quad & && (x \text{ and } \mathcal{X} \text{ are parametric variables being bound, and by (1) of (2.2)})\\
(2.14)\quad & \Leftrightarrow && \forall x\, (\, \exists \mathcal{X}\, \exists \mathcal{E}\, (F(\mathcal{X}, \mathcal{E}) \wedge A(\mathcal{X}, \mathcal{E})) \Rightarrow \forall \mathcal{X}\, \forall \mathcal{E}\, B(x, \mathcal{X}))\\
(2.15)\quad & \Leftrightarrow && \exists x\, \exists \mathcal{X}\, \exists \mathcal{E}\, (F(\mathcal{X}, \mathcal{E}) \wedge A(\mathcal{X}, \mathcal{E})) \Rightarrow \forall x\, \forall \mathcal{X}\, \forall \mathcal{E}\, B(x, \mathcal{X})\\
(2.16)\quad & \Leftrightarrow && \exists \mathcal{X}\, \exists \mathcal{E}\, (F(\mathcal{X}, \mathcal{E}) \wedge A(\mathcal{X}, \mathcal{E})) \Rightarrow \forall x\, \forall \mathcal{X}\, B(x, \mathcal{X})\\
(2.17)\quad & && (x \text{ in } \exists x\, \mathcal{X}\, \exists \mathcal{E}\, (F(\mathcal{X}, \mathcal{E}) \wedge A(\mathcal{X}, \mathcal{E})) \text{ and}\\
(2.18)\quad & && \qquad \mathcal{E} \text{ in } \forall x\, \forall \mathcal{X}\, \forall \mathcal{E}\, B(x, \mathcal{X}) \text{ are dummy variables being bound)}
\end{aligned}$$

where

$$\begin{aligned}
(2.19)\quad & && \forall x\, \forall \mathcal{X}\, B(x, \mathcal{X})\\
(2.20)\quad & \Leftrightarrow && \forall x\, \forall \mathcal{X}\, \exists N\, \forall n > N\, (x \in \mathcal{X}(n))\\
(2.21)\quad & \Leftrightarrow && \forall x\, \forall \mathcal{X}\, \exists N\, \forall n\, (n > N \Rightarrow x \in \mathcal{X}(n))\\
(2.22)\quad & \Leftrightarrow && \exists x\, \exists \mathcal{X}\, \forall N\, \exists n\, (n > N) \Rightarrow \forall x\, \forall \mathcal{X}\, \exists N\, \forall n\, (x \in \mathcal{X}(n))\\
(2.23)\quad & \Leftrightarrow && \forall N\, \exists n\, (n > N) \Rightarrow \forall x\, \forall \mathcal{X}\, \forall n\, (x \in \mathcal{X}(n)).
\end{aligned}$$

Now let us assume that $\exists \mathcal{Y}\, \forall m\, (\mathcal{Y}(m) \in \bar{\omega})$. Since for any x,

$$x \in \bar{\omega} \Leftrightarrow x \in \omega \wedge \forall \mathcal{X}\, \forall \mathcal{E} (F(\mathcal{X}, \mathcal{E}) \wedge A(\mathcal{X}, \mathcal{E}) \Rightarrow B(x, \mathcal{X})),$$

we have

$$
\begin{array}{lll}
(2.24) & & \exists\mathcal{Y}\,\forall m\,(\mathcal{Y}(m)\in\bar{\omega}) \\
(2.25) & \Leftrightarrow & \exists\mathcal{Y}\,\forall m\,(\mathcal{Y}(m)\in\omega\,\wedge\,\forall\mathcal{X}\,\forall\mathcal{E}(F(\mathcal{X},\mathcal{E})\wedge A(\mathcal{X},\mathcal{E})\,\Rightarrow\,B(\mathcal{Y}(m),\mathcal{X}))) \\
(2.26) & \Leftrightarrow & \exists\mathcal{Y}\,(\forall m(\mathcal{Y}(m)\in\omega)\,\wedge\,\forall m\forall\mathcal{X}\forall\mathcal{E}(F(\mathcal{X},\mathcal{E})\wedge A(\mathcal{X},\mathcal{E})\,\Rightarrow\,B(\mathcal{Y}(m),\mathcal{X}))) \\
(2.27) & \Rightarrow & \exists\mathcal{Y}\forall m(\mathcal{Y}(m)\in\omega)\,\wedge\,\exists\mathcal{Y}\forall m\forall\mathcal{X}\forall\mathcal{E}(F(\mathcal{X},\mathcal{E})\wedge A(\mathcal{X},\mathcal{E})\,\Rightarrow\,B(\mathcal{Y}(m),\mathcal{X}))
\end{array}
$$

where

$$
\begin{array}{lll}
(2.28) & & \exists\mathcal{Y}\,\forall m\,\forall\mathcal{X}\,\forall\mathcal{E}\,(F(\mathcal{X},\mathcal{E})\wedge A(\mathcal{X},\mathcal{E})\,\Rightarrow\,B(\mathcal{Y}(m),\mathcal{X})) \\
(2.29) & \Leftrightarrow & \exists\mathcal{Y}\,\forall m\,\forall\mathcal{X}\,(\,\exists\mathcal{E}(F(\mathcal{X},\mathcal{E})\wedge A(\mathcal{X},\mathcal{E}))\,\Rightarrow\,\forall\mathcal{E}\,B(\mathcal{Y}(m),\mathcal{X})) \\
(2.30) & \Leftrightarrow & \exists\mathcal{Y}\,(\,\exists m\,\exists\mathcal{X}\,\exists\mathcal{E}\,(F(\mathcal{X},\mathcal{E})\wedge A(\mathcal{X},\mathcal{E}))\,\Rightarrow\,\forall m\,\forall\mathcal{X}\,\forall\mathcal{E}\,B(\mathcal{Y}(m),\mathcal{X})) \\
(2.31) & \Leftrightarrow & \forall\mathcal{Y}\,\exists m\,\exists\mathcal{X}\,\exists\mathcal{E}\,(F(\mathcal{X},\mathcal{E})\wedge A(\mathcal{X},\mathcal{E}))\,\Rightarrow\,\exists\mathcal{Y}\,\forall m\,\forall\mathcal{X}\,\forall\mathcal{E}\,B(\mathcal{Y}(m),\mathcal{X}),
\end{array}
$$

in which

$$
\begin{array}{lll}
(2.32) & & \exists\mathcal{Y}\,\forall m\,\forall\mathcal{X}\,\forall\mathcal{E}\,B(\mathcal{Y}(m),\mathcal{X}) \\
(2.33) & \Leftrightarrow & \exists\mathcal{Y}\,\forall m\,\forall\mathcal{X}\,\forall\mathcal{E}\,\exists N\,\forall n\!>\!N(\mathcal{Y}(m)\in\mathcal{X}(n)) \\
(2.34) & \Leftrightarrow & \exists\mathcal{Y}\,\forall m\,\forall\mathcal{X}\,\forall\mathcal{E}\,\exists N\,\forall n\,(n>N\,\Rightarrow\,\mathcal{Y}(m)\in\mathcal{X}(n)) \\
(2.35) & \Leftrightarrow & \exists\mathcal{Y}\,\forall m\,\forall\mathcal{X}\,\exists N\,\forall n\,(n>N\,\Rightarrow\,\mathcal{Y}(m)\in\mathcal{X}(n)) \\
(2.36) & \Leftrightarrow & \forall\mathcal{Y}\,\exists m\,\exists\mathcal{X}\,\forall N\,\exists n\,(n>N)\,\Rightarrow\,\exists\mathcal{Y}\,\forall m\,\forall\mathcal{X}\,\exists N\,\forall n\,(\mathcal{Y}(m)\in\mathcal{X}(n)) \\
(2.37) & \Leftrightarrow & \forall N\,\exists n\,(n>N)\,\Rightarrow\,\exists\mathcal{Y}\,\forall m\,\forall\mathcal{X}\,\forall n\,(\mathcal{Y}(m)\in\mathcal{X}(n)).
\end{array}
$$

Then for some $\mathcal{Y}$ (which is introduced by the Lemma (2.1)), we can let $\mathcal{X}$ be $\mathcal{Y}$, and so we have

$$
\begin{array}{lll}
(2.38) & & \forall N\,\exists n\,(n>N)\,\Rightarrow\,\forall m\,\forall n\,(\mathcal{Y}(m)\in\mathcal{Y}(n)) \\
(2.39) & \Leftrightarrow & \forall N\,\exists m\,\exists n\,(n>N)\,\Rightarrow\,\exists N\,\forall m\,\forall n\,(\mathcal{Y}(m)\in\mathcal{Y}(n)) \\
(2.40) & \Leftrightarrow & \exists N\,\forall m\,\forall n\,(n>N\,\Rightarrow\,\mathcal{Y}(m)\in\mathcal{Y}(n)) \\
(2.41) & \Rightarrow & m\!>\!N\Rightarrow\mathcal{Y}(m)\in\mathcal{Y}(m)\text{ for some foregoing }N\text{ and any }m\text{ with }n\!=\!m \\
(2.42) & & (\,by\ \forall n\!\in\!\mathbb{N}\,\exists m\,(m=n)\,\wedge\,\forall m\,\exists n\!\in\!\mathbb{N}\,(n=m)),
\end{array}
$$

which is a contradiction because $\mathcal{Y}(m)\notin\mathcal{Y}(m)$ for such a $\mathcal{Y}$ and every $m\in\mathrm{dom}(\mathcal{Y})$, and so the assumption cannot hold. By the Law of Excluded Middle, if ZF is consistent then we have (2.1).
NB: Assuming $\exists\mathcal{Y}\,\forall m\,(\mathcal{Y}(m)\in\bar{\omega})$, by (2.35) – (2.37) and (2.38) – (2.40) we have for some $\mathcal{Y}$, $\forall m\,\exists N\,\forall n\!>\!N(\mathcal{Y}(m)\in\mathcal{Y}(n))\;\Leftrightarrow\;\exists N\,\forall m\,\forall n\!>\!N(\mathcal{Y}(m)\in\mathcal{Y}(n))$. □

From (2.1), and by

$$
\begin{array}{ll}
(2.43) & (1)\ \neg\,(\exists x\,Q(x))\,\equiv\,\forall x\,(\neg\,Q(x)), \\
 & (2)\ \neg\,(\forall x\,Q(x))\,\equiv\,\exists x\,(\neg\,Q(x)),
\end{array}
$$

we have

$$
\begin{array}{lll}
(2.44)\ \neg\,(\,\exists\mathcal{Y}\,\forall m\,(\mathcal{Y}(m)\in\bar{\omega})) & \Leftrightarrow & \forall\mathcal{Y}\,(\,\neg\,(\forall m\,(\mathcal{Y}(m)\in\bar{\omega}))) \\
(2.45) & \Leftrightarrow & \forall\mathcal{Y}\,\exists m\,(\neg\,\mathcal{Y}(m)\in\bar{\omega}) \\
(2.46) & \Leftrightarrow & \forall\mathcal{Y}\,\exists m\,(\mathcal{Y}(m)\notin\bar{\omega}).
\end{array}
$$

Lemma 2.2. *Let $\mathcal{Y}$ be the variable function such that $\mathcal{Y} \to \infty$ where*
$$\mathrm{dom}(\mathcal{Y}) = \mathcal{N} \textit{ and } \mathrm{ran}(\mathcal{Y}) \subset \omega,$$
and let m be the variable object such that $m \in \mathrm{dom}(\mathcal{Y})$. Then
$$\neg\,(\,\forall\mathcal{Y}\,\forall m\,(\mathcal{Y}(m) \in \bar{\omega})). \tag{2.47}$$

Proof. Assume that $\forall\mathcal{Y}\,\forall m\,(\mathcal{Y}(m) \in \bar{\omega})$. Similar to (2.24)–(2.37), we have
$$\forall N\,\exists n\,(n > N) \;\Rightarrow\; \forall\mathcal{Y}\,\forall m\,\forall\mathcal{X}\,\forall n\,(\mathcal{Y}(m) \in \mathcal{X}(n)).$$
Then for any of the foregoing $\mathcal{Y}$s, we can let $\mathcal{X}$ be $\mathcal{Y}$, and so we obtain:
$$\forall N\,\exists n\,(n > N) \;\Rightarrow\; \forall m\,\forall n\,(\mathcal{Y}(m) \in \mathcal{Y}(n)).$$
By (2.38) – (2.42), there is a contradiction in it. Therefore the assumption cannot hold, and so by the Law of Excluded Middle we have (2.47). □

From (2.47), by (2) of (2.43), we have
$$\begin{aligned}
&(2.48)\quad \neg\,(\,\forall\mathcal{Y}\,\forall m\,(\mathcal{Y}(m) \in \bar{\omega})) &&\Leftrightarrow\; \exists\mathcal{Y}\,(\neg\,(\,\forall m\,(\mathcal{Y}(m) \in \bar{\omega})))\\
&(2.49) &&\Leftrightarrow\; \exists\mathcal{Y}\,\exists m\,(\neg\,\mathcal{Y}(m) \in \bar{\omega})\\
&(2.50) &&\Leftrightarrow\; \exists\mathcal{Y}\,\exists m\,(\mathcal{Y}(m) \notin \bar{\omega}).
\end{aligned}$$

In addition, by (1.1) – (1.5) and (2.43), we can obtain

(2.51) (1) $\exists\mathcal{Y}\,\exists m\,(\mathcal{Y}(m) \in \bar{\omega})$;

(2) $\neg\,(\,\exists\mathcal{Y}\,\forall m\,(\mathcal{Y}(m) \notin \bar{\omega}))$.

Combining the results with (2.43) we obtain
$$\exists\mathcal{Y}\,\exists m\,(\mathcal{Y}(m) \in \bar{\omega}) \;\Leftrightarrow\; \neg\,(\,\forall\mathcal{Y}\,\forall m\,(\mathcal{Y}(m) \notin \bar{\omega})); \tag{2.52}$$
$$\neg\,(\,\exists\mathcal{Y}\,\forall m\,(\mathcal{Y}(m) \notin \bar{\omega})) \;\Leftrightarrow\; \forall\mathcal{Y}\,\exists m\,(\mathcal{Y}(m) \in \bar{\omega}). \tag{2.53}$$

Thus, if $\mathcal{Y}$ is the variable function such that $\mathcal{Y} \to \infty$ where $\mathrm{dom}(\mathcal{Y}) = \mathcal{N}$ and $\mathrm{ran}(\mathcal{Y}) \subset \omega$, and m is the variable object such that $m \in \mathrm{dom}(\mathcal{Y})$, then each of the following cannot hold.

(2.54) (1) $\exists\mathcal{Y}\,\forall m\,(\mathcal{Y}(m) \in \bar{\omega})$;

(2) $\forall\mathcal{Y}\,\forall m\,(\mathcal{Y}(m) \in \bar{\omega})$;

(3) $\exists\mathcal{Y}\,\forall m\,(\mathcal{Y}(m) \notin \bar{\omega})$;

(4) $\forall\mathcal{Y}\,\forall m\,(\mathcal{Y}(m) \notin \bar{\omega})$.

By (1.1) – (1.3), $\mathcal{Y}(m) \in \omega$ holds for any of the foregoing $\mathcal{Y}$s and any $m \in \mathrm{dom}(\mathcal{Y})$, and so by (2.1) and (2.47) we obtain:

Theorem 2.1.
$$\bar{\omega} \neq \omega. \tag{2.55}$$

By (1.5) and (2.55), we clearly have $\omega - \bar{\omega} \neq \varnothing$ and $\omega - \bar{\omega} \neq \omega$.

Theorem 2.2.
$$\forall\alpha\in\bar{\omega}\,\exists\Theta\in\omega\,(\alpha \in \Theta). \tag{2.56}$$

Proof. By (1.1) – (1.4), for any α,

$$\begin{aligned}
&(2.57) && \alpha \in \bar{\omega} && \Leftrightarrow && \alpha \in \omega \wedge \forall \mathcal{X} \forall \mathcal{E} \, (F(\mathcal{X},\mathcal{E}) \wedge A(\mathcal{X},\mathcal{E}) \Rightarrow B(\alpha,\mathcal{X}))\\
&(2.58) && && \Rightarrow && \alpha \in \omega \wedge \exists \mathcal{X} \, \exists N \, \exists n > N \, (\mathcal{X}(n) \in \omega \wedge \alpha \in \mathcal{X}(n))\\
&(2.59) && && \Rightarrow && \exists \mathcal{X} \, \exists N \, \exists n > N \, (\alpha \in \omega \wedge \mathcal{X}(n) \in \omega \wedge \alpha \in \mathcal{X}(n))\\
&(2.60) && && && (\textit{because } \alpha \textit{ is a parametric variable being bound and}\\
&(2.61) && && && \quad \textit{by the property that } q \wedge \exists x S(x) \equiv \exists x (q \wedge S(x)))\\
&(2.62) && && \Rightarrow && \exists \Theta \, (\Theta \in \omega \wedge \alpha \in \Theta)\\
&(2.63) && && \Leftrightarrow && \exists \Theta \in \omega \, (\alpha \in \Theta).
\end{aligned}$$

Or, put more briefly, for any α,

$$\begin{aligned}
&(2.64) && \alpha \in \bar{\omega} && \Rightarrow && \alpha \in \omega\\
&(2.65) && && \Leftrightarrow && \alpha \in \bigcup \omega\\
&(2.66) && && \Leftrightarrow && \exists \Theta \, (\alpha \in \Theta \wedge \Theta \in \omega)\\
&(2.67) && && \Leftrightarrow && \exists \Theta \in \omega \, (\alpha \in \Theta).
\end{aligned}$$

Hence we have (2.56). □

Theorem 2.3.

$$(2.68) \qquad \neg \, (\forall \Theta \in \omega \, (\Theta \subset \bar{\omega})).$$

Proof. Assume that $\forall \Theta \in \omega \, (\Theta \subset \bar{\omega})$. Since for any α,

$$\begin{aligned}
&(2.69) && \alpha \in \omega && \Leftrightarrow && \alpha \in \bigcup \omega\\
&(2.70) && && \Leftrightarrow && \exists \Theta \, (\alpha \in \Theta \wedge \Theta \in \omega)\\
&(2.71) && && \Rightarrow && \exists \Theta \, (\alpha \in \Theta \wedge \Theta \subset \bar{\omega})\\
&(2.72) && && \Rightarrow && \alpha \in \bar{\omega},
\end{aligned}$$

we have $\omega \subset \bar{\omega}$. And by (1.5), we would have $\bar{\omega} = \omega$, which is contradictory with (2.55). Thus we have (2.68). □

From (2.68), by (2) of (2.43), we have

$$\begin{aligned}
&(2.73) && \neg \, (\forall \Theta \in \omega \, (\Theta \subset \bar{\omega})) && \Leftrightarrow && \exists \Theta \in \omega \, (\neg (\Theta \subset \bar{\omega}))\\
&(2.74) && && \Leftrightarrow && \exists \Theta \in \omega \, (\Theta \not\subset \bar{\omega}).
\end{aligned}$$

Compare $\bar{\omega}$ with ω:

$$\begin{aligned}
&(2.75) && \forall \alpha \, (\alpha \in \bar{\omega} \Rightarrow \exists \Theta \, (\alpha \in \Theta \wedge \Theta \in \omega));\\
&(2.76) && \forall \alpha \, (\alpha \in \omega \Leftrightarrow \exists \Theta \, (\alpha \in \Theta \wedge \Theta \in \omega)).
\end{aligned}$$

Theorem 2.4.

$$(2.77) \qquad \sup \bar{\omega} = \bigcup \bar{\omega} = \bar{\omega}.$$

Proof. Firstly, for any α,

$$\begin{aligned}
&(2.78) && && \alpha \in \bigcup \bar{\omega}\\
&(2.79) && \Leftrightarrow && \exists \vartheta \, (\alpha \in \vartheta \wedge \vartheta \in \bar{\omega})\\
&(2.80) && \Leftrightarrow && \exists \vartheta \, (\alpha \in \vartheta \wedge \vartheta \in \omega \wedge \forall \mathcal{X} \forall \mathcal{E} \, (F(\mathcal{X},\mathcal{E}) \wedge A(\mathcal{X},\mathcal{E}) \Rightarrow B(\vartheta,\mathcal{X})))\\
&(2.81) && \Leftrightarrow && \exists \vartheta \, (\alpha \in \vartheta \wedge \vartheta \in \omega \wedge \alpha \in \vartheta \wedge \forall \mathcal{X} \forall \mathcal{E} \, (F(\mathcal{X},\mathcal{E}) \wedge A(\mathcal{X},\mathcal{E}) \Rightarrow B(\vartheta,\mathcal{X})))\\
&(2.82) && \Rightarrow && \alpha \in \omega \wedge \exists \vartheta \, (\alpha \in \vartheta \wedge \forall \mathcal{X} \forall \mathcal{E} \, (F(\mathcal{X},\mathcal{E}) \wedge A(\mathcal{X},\mathcal{E}) \Rightarrow B(\vartheta,\mathcal{X}))),
\end{aligned}$$

where for any $\mathcal{X}$, $\mathcal{E}$, if $F(\mathcal{X},\mathcal{E})$ and $A(\mathcal{X},\mathcal{E})$ hold then we have for any $\vartheta \in \bar{\omega}$,

$$\begin{aligned}
&(2.83) && B(\vartheta,\mathcal{X})\\
&(2.84) &\Rightarrow\ & \exists N\,\forall n\!>\!N\,(\vartheta\in\mathcal{X}(n))\\
&(2.85) &\Rightarrow\ & \exists N\,\forall n\,(n>N\ \Rightarrow\ \vartheta\in\mathcal{X}(n))\\
&(2.86) &\Rightarrow\ & \forall N\,\exists n\,(n>N)\ \Rightarrow\ \exists N\,\forall n\,(\vartheta\in\mathcal{X}(n))\\
&(2.87) &\Rightarrow\ & \forall N\,\exists n\,(n>N)\ \Rightarrow\ \forall n\,(\vartheta\in\mathcal{X}(n))\\
&(2.88) &\Rightarrow\ & \forall N\,\exists n\,(n>N)\ \Rightarrow\ \forall n\,(\vartheta\subset\mathcal{X}(n))\\
&(2.89) &\Rightarrow\ & \forall N\,\exists n\,(n>N)\ \Rightarrow\ \forall n\,\forall\beta\,(\beta\in\vartheta\ \Rightarrow\ \beta\in\mathcal{X}(n))\\
&(2.90) &\Rightarrow\ & \exists N\,\forall n\,\forall\beta\,(n>N\ \Rightarrow\ (\beta\in\vartheta\ \Rightarrow\ \beta\in\mathcal{X}(n))),
\end{aligned}$$

and therefore

$$\begin{aligned}
&(2.91) && \exists\vartheta\,(\alpha\in\vartheta\ \wedge\ \forall\mathcal{X}\,\forall\mathcal{E}\,(F(\mathcal{X},\mathcal{E})\wedge A(\mathcal{X},\mathcal{E})\ \Rightarrow\ B(\vartheta,\mathcal{X})))\\
&(2.92) &\Rightarrow\ & \forall\mathcal{X}\,\forall\mathcal{E}\,(F(\mathcal{X},\mathcal{E})\wedge A(\mathcal{X},\mathcal{E})\ \Rightarrow\ B(\alpha,\mathcal{X}));
\end{aligned}$$

hence we have

$$\begin{aligned}
&(2.93) && \alpha\in\bigcup\bar{\omega}\ \Rightarrow\ \alpha\in\omega\ \wedge\ \forall\mathcal{X}\,\forall\mathcal{E}\,(F(\mathcal{X},\mathcal{E})\wedge A(\mathcal{X},\mathcal{E})\Rightarrow B(\alpha,\mathcal{X}))\\
&(2.94) && \qquad\qquad\ \Rightarrow\ \alpha\in\bar{\omega}.
\end{aligned}$$

Then it follows that

$$\bigcup\bar{\omega}\ \subset\ \bar{\omega}. \tag{2.95}$$

Secondly, by (1.2) – (1.4), for any $\alpha\in\bar{\omega}$ we have

$$\begin{aligned}
&(2.96) && \forall\mathcal{X}'\,\forall\mathcal{E}'\,(F(\mathcal{X}',\mathcal{E}')\wedge A(\mathcal{X}',\mathcal{E}')\ \Rightarrow\ B(\alpha,\mathcal{X}'))\\
&(2.97) &\Leftrightarrow\ & \forall\mathcal{X}'\,\forall\mathcal{E}'\,(F(\mathcal{X}',\mathcal{E}')\wedge A(\mathcal{X}',\mathcal{E}')\ \Rightarrow\ \exists N\,\forall n\!>\!N\,(\alpha\in\mathcal{X}'(n))\\
&(2.98) &\Leftrightarrow\ & \forall\mathcal{X}'\,\forall\mathcal{E}'\,(F(\mathcal{X}',\mathcal{E}')\wedge A(\mathcal{X}',\mathcal{E}')\ \Rightarrow\ \exists N\,\forall n\,(n>N\ \Rightarrow\ \alpha\in\mathcal{X}'(n)),
\end{aligned}$$

and so for any such function $\mathcal{X}'$ with some N there is some $N_\alpha > N$ such that

$$\alpha\ \in\ \mathcal{X}'(N_\alpha). \tag{2.99}$$

Since $N_\alpha\in\mathbb{N}$ and $0<\mathcal{X}'(N_\alpha)\in\omega$, by (1.2) and (1.3), for any $\mathcal{X}''$ with some $\mathcal{E}''$ we have $\exists M=\mathcal{E}''(\mathcal{X}'(N_\alpha))\,\forall n\!>\!M\,(\mathcal{X}''(n)>\mathcal{X}'(N_\alpha))$ if $F(\mathcal{X}'',\mathcal{E}'')$ and $A(\mathcal{X}'',\mathcal{E}'')$ hold, and so for any such $\mathcal{X}''$ there is some M such that

$$\forall n\!>\!M\ (\mathcal{X}'(N_\alpha)\in\mathcal{X}''(n)); \tag{2.100}$$

thus by (1.4), we have $\mathcal{X}'(N_\alpha)\ \in\ \bar{\omega}$. Therefore

$$\forall\alpha\!\in\!\bar{\omega}\,\exists\vartheta\!\in\!\bar{\omega}\,(\alpha\in\vartheta), \tag{2.101}$$

that is, $\bar{\omega}$ has no the maximal element, and hence

$$\sup\bar{\omega}\ =\ \bigcup\bar{\omega}. \tag{2.102}$$

Combining (2.95) with (2.102), it follows that (2.77). $\square$

Since $\bar{\omega}\neq\varnothing$ is a set of finite ordinals, $\bigcup\bar{\omega}$ is an ordinal. Rather oddly enough, by (2.77) and (2.55), $\bar{\omega}=\bigcup\bar{\omega}$ is exactly a limit ordinal, as distinguished from the limit ordinal ω, and thus we have

Corollary 2.1.

(2.103) (1) *$\forall\alpha\,(\alpha\in\bar{\omega} \Leftrightarrow \exists\vartheta\,(\alpha\in\vartheta\wedge\vartheta\in\bar{\omega}))$;*

(2) *$\forall\alpha\in\bar{\omega}\,(\alpha\subset\bar{\omega})$;*

(3) *Let $\mathcal{P}(\bar{\omega})$ be the power set of $\bar{\omega}$. Then $\bar{\omega}\subset\mathcal{P}(\bar{\omega})$, i. e., $\forall\alpha\in\bar{\omega}\,(\alpha\in\mathcal{P}(\bar{\omega}))$.*

In particular, $U\subset\omega$ is cofinal in ω if $\sup U=\omega$, and $\sup\bar{\omega}=\bar{\omega}\neq\omega$, hence the set $\bar{\omega}$ cannot be cofinal in ω, i. e., $\operatorname{cof}(\omega,\bar{\omega})$ does not hold. This also shows that in ZF the limit of a sequence based on (1.2) and (1.3) will be fundamentally different than the limit of a sequence based on the Axiom of Infinity. However, the natural question to ask is, whether $\bar{\omega}\in\omega$ or not. Obviously, from (1.1) it follows that

Theorem 2.5.

$$\bar{\omega}\notin\omega. \tag{2.104}$$

Theorem 2.6.

(2.105) *$\bar{\omega}$ is infinite but is D-finite.*

Proof. On the one hand, by (2.101), there is no a one-to-one mapping of $\bar{\omega}$ onto a natural number, that is, $\bar{\omega}$ is infinite.

On the other hand, if we let $\mathcal{T}$ be an arbitrary proper subset of $\bar{\omega}$, then $\mathcal{T}\subset\bar{\omega}$ and $\mathcal{T}\neq\bar{\omega}$, and thus $\bigcup\mathcal{T}\subset\bigcup\bar{\omega}$. Obviously, $\bigcup\mathcal{T}$ and $\bigcup\bar{\omega}$ are ordinals, and we have:

(1) If $\bigcup\mathcal{T}\neq\bigcup\bar{\omega}$, then since $\bigcup\bar{\omega}=\bar{\omega}$ is a limit ordinal and $\bigcup\mathcal{T}\subsetneq\bar{\omega}$, $\bigcup\mathcal{T}$ is not a limit ordinal, and so $\mathcal{T}$ has the maximal element; hence there is no a one-to-one mapping of $\bar{\omega}$ onto such a proper subset $\mathcal{T}$ of $\bar{\omega}$.

(2) If $\bigcup\mathcal{T}=\bigcup\bar{\omega}$, then $\bigcup\mathcal{T}=\bar{\omega}$ is a limit ordinal. Now we let $\mathcal{Z}\to\infty$ be such that $\operatorname{dom}(\mathcal{Z})=\mathcal{N}$ and $\operatorname{ran}(\mathcal{Z})=\omega$, and let m be a variable such that $m\in\operatorname{dom}(\mathcal{Z})$. By (2.46) and (2.74), there is some m such that $\mathcal{Z}(m)\in\omega$ and $\mathcal{Z}(m)\notin\bar{\omega}$, and so if we let $\mathfrak{M}$ be the smallest such m, then $\mathcal{Z}(\mathfrak{M})\in\omega\wedge\mathcal{Z}(\mathfrak{M})\notin\bar{\omega}$, and by (1.2) − (1.4) and (2.77), we have $\bar{\omega}=\{\mathcal{Z}(m)\in\omega : m\in\operatorname{dom}(\mathcal{Z})\wedge m<\mathfrak{M}\}$, where $\mathfrak{M}\in\mathcal{N}$ will be large enough, at least not less than any $\alpha\in\bar{\omega}$; thus $\bar{\omega}$ has $\mathfrak{M}-1$ elements even though $\mathfrak{M}$ is uncertain. Assume that there exists a one-to-one mapping of $\bar{\omega}$ onto the proper subset $\mathcal{T}$ of $\bar{\omega}$ where $\mathcal{T}\subsetneq\bar{\omega}$ and $\bigcup\mathcal{T}=\bar{\omega}$. Then since there is at least one $i\in\omega$ such that $\mathcal{Z}(\mathfrak{M}-i)\notin\mathcal{T}$, $\mathcal{T}$ has at most $\mathfrak{M}-2$ elements, and so $\mathfrak{M}-1=\mathfrak{M}-2$, that is, $1=2$, a contradiction; hence there is no a one-to-one mapping of $\bar{\omega}$ onto the proper subset $\mathcal{T}$ of $\bar{\omega}$.

In a word, there is no a one-to-one mapping of $\bar{\omega}$ onto a proper subset of $\bar{\omega}$, that is, $\bar{\omega}$ is Dedekind-finite (D-finite). □

Since $\bar{\omega}$ is infinite, if we consider $\mathcal{U}=\{s\subset\bar{\omega} : s \text{ is finite}\}$, then it follows that $\bar{\omega}$ is Tarski-infinite (T-infinite). By (2.105), $\bar{\omega}$ is not just T-infinite, but also D-finite; hence it does not satisfy the Axiom of Choice. This result would not be inconsistent in ZF, because one cannot prove that every D-finite set is finite without the Axiom of Choice.[4]

By ZF, it is common knowledge that $|\omega|=\omega=\omega_0=\aleph_0$ and $|\alpha|=\alpha$ for every $\alpha<\omega$. Now we consider the cardinality $|\bar{\omega}|$ of $\bar{\omega}$, we first notice that if for any ordinal α, β such that $\beta\leq\alpha$ and $|\beta|=|\alpha|$, there exists an ordinal number $\beta_0\in\beta^+$ such that $\operatorname{cof}(\alpha,\beta_0)$ holds. Since $\operatorname{cof}(\omega,\bar{\omega})$ does not hold, we have $|\bar{\omega}|\neq|\omega|$.

Theorem 2.7.

$$\exists \alpha \in \omega\, (\alpha < |\bar{\omega}| < |\omega|). \tag{2.106}$$

Proof. On the one hand, by (2.77), $\bar{\omega}$ is transitive and has no maximal element, and so by (2) of (2.103) $\forall \alpha \in \bar{\omega}\, (\alpha < |\bar{\omega}|)$ holds. And by (1.4), we have

$$\exists \alpha \in \omega\, (\alpha < |\bar{\omega}|). \tag{2.107}$$

On the other hand, by (1.5), we have $|\bar{\omega}| \leq |\omega|$. Assume that $|\bar{\omega}| = |\omega|$. Then there exists a one-to-one mapping of ω onto $\bar{\omega}$. Let h be a one-to-one mapping of ω onto $\bar{\omega}$ such that $h(\alpha) = \vartheta$. Then we have a one-to-one mapping $2\alpha \mapsto 2\vartheta$ of the proper subset $\{2\alpha : \alpha \in \omega\}$ of ω onto the proper subset $\{2\vartheta : \vartheta \in \bar{\omega}\}$ of $\bar{\omega}$. Since the mapping $\alpha \mapsto 2\alpha$ of ω onto $\{2\alpha : \alpha \in \omega\}$ is also one-to-one, we have $|\{2\vartheta : \vartheta \in \bar{\omega}\}| = |\bar{\omega}|$. However, this is impossible since $|\bar{\omega}|$ is D-finite. Therefore,

$$|\bar{\omega}| < |\omega|. \tag{2.108}$$

By (2.107) and (2.108), we have (2.106). □

Theorem 2.8.

$$2^{|\bar{\omega}|} \leq |\omega|. \tag{2.109}$$

Proof. By the Power Set and Separation Axioms, we let $\mathcal{P}(\bar{\omega}) = \{s : s \subset \bar{\omega}\}$,

$$\mathcal{P}_D(\omega) = \{s \in \mathcal{P}(\omega) : |s| < |\omega| \wedge s \text{ is D-finite}\}$$

and $\mathcal{P}_{<\omega_0}(\omega) = \{s \in \mathcal{P}(\omega) : |s| < |\omega|\}$. Since $\mathcal{P}(\bar{\omega}) \subset \mathcal{P}_D(\omega)$ (by (2.105) – (2.108)) and $\mathcal{P}_D(\omega) \subset \mathcal{P}_{<\omega_0}(\omega)$, we have

$$|\mathcal{P}(\bar{\omega})| \leq |\mathcal{P}_{<\omega_0}(\omega)|. \tag{2.110}$$

Subsequently, let us consider the cardinality of $\mathcal{P}_{<\omega_0}(\omega)$. Given a $\mathcal{S} \in \mathcal{P}_{<\omega_0}(\omega)$, we construct a function $\varphi_{\mathcal{S}} : \omega \to \{0, 1\}$ such that $\varphi_{\mathcal{S}}(i) = a_i$, where $a_i = 1$ if $i \in \mathcal{S}$, and $a_i = 0$ otherwise; hence

$$\mathcal{S} = \{i \in \omega : \mathcal{I}(\mathcal{S}) \in \omega \wedge i \leq \mathcal{I}(\mathcal{S}) \wedge \varphi_{\mathcal{S}}(i) = 1\}.$$

If $r_{\mathcal{S}} = a_{\mathcal{I}(\mathcal{S})} a_{\mathcal{I}(\mathcal{S})-1} \ldots a_i \ldots a_2 a_1 a_0$ such that

$$\mathcal{I}(\mathcal{S}) \in \omega \wedge 0 \leq i \leq \mathcal{I}(\mathcal{S}) \wedge a_i \in \{0, 1\},$$

then

$$r_{\mathcal{S}} \in \omega \wedge r_{\mathcal{S}} = \varphi_{\mathcal{S}}(\mathcal{I}(\mathcal{S}))\varphi_{\mathcal{S}}(\mathcal{I}(\mathcal{S}) - 1) \ldots \varphi_{\mathcal{S}}(i) \ldots \varphi_{\mathcal{S}}(2)\varphi_{\mathcal{S}}(1)\varphi_{\mathcal{S}}(0),$$

and so for every $\mathcal{S}_\alpha$, $\mathcal{S}_\beta \in \mathcal{P}_{<\omega_0}(\omega)$,

$$\mathcal{S}_\alpha \neq \mathcal{S}_\beta \text{ if and only if } r_{\mathcal{S}_\alpha} \neq r_{\mathcal{S}_\beta}.$$

It is clear that there is a one-to-one mapping of $\mathcal{P}_{<\omega_0}(\omega)$ into ω. Thus

$$|\mathcal{P}_{<\omega_0}(\omega)| \leq |\omega|. \tag{2.111}$$

Then, since $|\mathcal{P}(\bar{\omega})| = 2^{|\bar{\omega}|}$, by (2.110) and (2.111) we have (2.109). □

Theorem 2.9.

(2.112) (1) $\mathrm{cf}\,(|\bar{\omega}|) = |\bar{\omega}|$;

(2) $\omega_1^{|\bar{\omega}|} = \omega_1$.

Proof. (1) Here we let $|\bar{\omega}| = \bar{\omega}$. By (2.106), we can consider the cofinality of $|\bar{\omega}|$. Let $\beta < \omega$ be a finite ordinal such that $\operatorname{cof}(|\bar{\omega}|, \beta)$ holds. Then there exists an ordinal number β_1 such that $\beta = \beta_1^+$, and there is a function ψ such that $\operatorname{cof}(|\bar{\omega}|, \beta)$ holds. Let $\alpha_1 \in |\bar{\omega}|$ be such that $\psi(\beta_1) = \alpha_1$. Then there is an ordinal number $\alpha_2 \in |\bar{\omega}|$ such that $\alpha_1 < \alpha_2$, and thus there is an ordinal number $\beta_2 \in \beta$ such that $\psi(\beta_2) = \alpha_2$. However, it is impossible since $\beta_2 \leq \beta_1$ and ψ is strictly monotonically increasing. Hence we only have $\beta = |\bar{\omega}|$, i. e., $\operatorname{cof}(|\bar{\omega}|, |\bar{\omega}|)$ and $\operatorname{cf}(|\bar{\omega}|) = |\bar{\omega}|$.

(2) By (2.106), there is some $\alpha < \omega$ such that $\omega_1^{|\alpha|} \leq \omega_1^{|\bar{\omega}|}$ and $\omega_1^{|\bar{\omega}|} \leq \omega_1^{|\omega|}$, and since $\omega_1^{|\alpha|} = \omega_1$ and $\omega_1^{|\omega|} = \omega_0^{\omega_0} \cdot \omega_1 = 2^{\omega_0}$, we have

$$\omega_1 \leq \omega_1^{|\bar{\omega}|} \leq 2^{\omega_0}. \tag{2.113}$$

By (2.106) and (2.109), we clearly have $|\bar{\omega}| < \omega_1$ and $2^{|\bar{\omega}|} = |\bar{\omega}|^{|\bar{\omega}|} \leq |\omega|$, and since ω_1 is a regular and successor cardinal, by the Hausdorff formula and the cardinal arithmetic we have

$$\begin{aligned}
\omega_1^{|\bar{\omega}|} &= \omega_0^{|\bar{\omega}|} \cdot \omega_1 && (2.114)\\
&= \Big(\sum_{\alpha<\omega_0} |\alpha|^{|\bar{\omega}|}\Big) \cdot \omega_1 && (2.115)\\
&= 2^{|\bar{\omega}|} \cdot \omega_0 \cdot \omega_1 && (2.116)\\
&\leq \omega_0 \cdot \omega_0 \cdot \omega_1 && (2.117)\\
&= \omega_1. && (2.118)
\end{aligned}$$

Combining (2.113) and (2.114) – (2.118), we conclude that $\omega_1^{|\bar{\omega}|} = \omega_1$ holds. $\square$

It is peculiar that there exists the infinite subset $\bar{\omega}$ of the countable ω that is not countable but is D-finite and can asymptotically converge to the countable ω. This will not lead to contradiction under ZF unless ZF is inconsistent.

3. FORCING METHOD AND ZF WITH $\bar{\omega}$

Theorem 3.1. *If* ZF *is consistent, then* $\bar{\omega}$ *and* ω *are indistinguishable in the forcing method.*

Proof. By (2.106) and (2.112), it is clear that $|\bar{\omega}| < \omega_0$, $\operatorname{cf}(|\bar{\omega}|) = |\bar{\omega}|$ and $\omega_1^{|\bar{\omega}|} = \omega_1$. Thereby, we let P be the set of all functions p such that

(3.1) (i) $\operatorname{dom}(p)$ is a finite subset of $\omega_1 \times \bar{\omega}$,
(ii) $\operatorname{ran}(p) \subset \{0, 1\}$,

and let p be stronger than q if and only if $p \supset q$.

Then the sets $\mathcal{D}_{\xi,n} = \{p \in P : <\xi, n> \in \operatorname{dom}(p)\}$ are dense in P. Let

$$\mathfrak{D} = \{\mathcal{D}_{\xi,n} : \xi \in \omega_1 \wedge n \in \bar{\omega}\}.$$

By Martin's Axiom $\mathrm{MA}_{\aleph_1}$, there exists a $\mathfrak{D}$-generic filter on P.

If G is a generic set of conditions, we let $f = \bigcup G$. We claim that

(3.2) (i) f is a function;
(ii) $\operatorname{dom}(f) = \omega_1 \times \bar{\omega}$.

Part (i) of (3.2) holds because G is a filter. For part (ii), G is a $\mathfrak{D}$-generic filter on P, hence G meets each set in $\mathfrak{D}$, and so $<\xi, n> \in \operatorname{dom}(f)$ for all $<\xi, n> \in \omega_1 \times \bar{\omega}$.

Now, for each $\xi < \omega_1$, let $f_\xi : \bar{\omega} \to \{0, 1\}$ be the function defined as follows:

$$f_\xi(n) = f(\xi, n).$$

If $\xi \neq \zeta$, then $f_\xi \neq f_\zeta$; this is because the set

$$\mathcal{D} = \{p \in P : p(\xi, n) \neq p(\zeta, n) \text{ for some } n\}$$

is dense in P and hence $G \cap \mathcal{D} \neq \varnothing$. Thus in $V[G]$ we have a one-to-one mapping $\xi \mapsto f_\xi$ of ω_1 into $\{0,1\}^{\bar{\omega}}$.

Since each f_ξ is the characteristic function of a set $a_\xi \subset \bar{\omega}$, there is a generic extension $V[G]$ such that

$$2^{|\bar{\omega}|} > \omega_0. \tag{3.3}$$

But by (2.109), and using the Cantor's Theorem, there can be only

$$|\bar{\omega}| < 2^{|\bar{\omega}|} \leq \omega_0 \ \text{ and } \ 2^{|\omega|} > |\omega| = \omega_0,$$

which is contradictory with the conclusion (3.3). Hence, either ZF is inconsistent, or $\bar{\omega}$ and ω are indistinguishable in the forcing method. □

Thus, if we believe that the method of forcing must be a perfect method, then ZF seems to be precisely inconsistent. "Apropos of the open question of consistency Poincaré remarked, 'We have put a fence around the herd to protect it from the wolves but we do not know whether some wolves were not already within the fence."'[2]

And in ZF, the set ω claims that the natural numbers will certainly belong to it to Infinity, but the set $\bar{\omega}$ claims that the natural numbers must not belong to it to Infinity. Just as, in geometry, the projective line claims that the points would lie in it at Infinity, but the Euclidean line claims that the points cannot lie in it at Infinity. Of course, if the infinite ω can be considered as a set, then its subclass $\bar{\omega}$ is exactly the T-infinite and D-finite set asymptotic to it, and this means that $\bar{\omega}$ will never be able to reach ω. In any case, $\bar{\omega}$ is indeed a set that differs from ω by ZF, otherwise ZF is inconsistent. So we have to be very, very careful when the extrapolations from finiteness to infiniteness are considered.

4. Concluding Remarks

By ZF, $\bar{\omega}$ is a non-empty proper subset of ω and it is asymptotic with respect to ω. If ZF is consistent, then the axioms of ZF and the fact that there exists the set $\bar{\omega}$ are also consistent; otherwise, ZF itself is inconsistent. Meanwhile, If ZF is consistent, then ω and $\bar{\omega}$ are exactly indistinguishable in the forcing method, and thus the description power of the forcing method is inadequate for ZF; conversely, if the forcing method does not try to discriminate between $\bar{\omega}$ and ω, then it would be inconsistent with ZF. Therefore, some related problems such as the consistency of ZF, the adequacy of the forcing method and CH independent of ZF, are needed to further study in ZF + $\bar{\omega} \neq \omega$ (i. e. ZF $\cup \{\bar{\omega} \neq \omega\}$), although $\bar{\omega} \neq \omega$ is merely an inference of ZF. But first, a very basic question is, what happens if the fields of the functions of (1.2) and (1.3) are ω_1?

In particular, there exists the T-infinite and D-finite subset $\bar{\omega}$ of ω that can not be put into one-to-one correspondence with a finite ordinal and can not be put into one-to-one correspondence with the infinite ω under ZF; nevertheless, the cardinality of such a set $\bar{\omega}$ is merely a measurement of the boundary of all finite ordinals, and hence there are no cardinals between the D-finite and the countable. In other words, there is an infinite set between the finite and the countable but it is unable to transcend all D-finite sets until the countable. Isn't that extraordinary? What if the set of all real numbers is also in a similar state of subsets?

Problem 1: Is there an infinite set of real numbers that could not be put into one-to-one correspondence with the natural numbers and could not be put into one-to-one correspondence with the real numbers?[3]

Problem 2: Is it the case that there are no cardinals between the countable and the continuum?[5]

By analogy with the case of $\bar{\omega}$, on the Problem 1 it should follow that $\neg$CH holds, but on the Problem 2 it should follow that CH holds. Thus the Problem 1 and the Problem 2 would not be equivalent to one another in ZF, are they so?

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