Bimeromorphic Equivalence: Taking Compact Ricci-Flat Kahler Geometry with an Extended Relation Between Morita Equivalence and Fujiki's Class C Manifold Taking Twisted K–Theory [New Version]

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December 13, 2023

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There's a relation to this aspect of homological algebra in the bimeromorphic equivalent between Calabi-Yau and Fukaya Category with furthermore connectivity in Fujiki's Class C Category. It is commonly known that Shift Operators play a nontrivial role in modelling the Abelian and the A_∞ where the derived category (as there's many equivalence in case of the triangulated category with derived categories) where a specific relation to K−Polystability can be found in case of a peculiar FANO surface which is on one hand associated with higher order J-Holomorphic Polygons and there's the relation of HOMOLOGICAL MIRROR SYMMETRY (connecting the equivalence to Analytical with Algebro-Geometric Model). This helps in the development of A and B-Model and their relation in Supersymmetric string Theory. Any Fukaya Category for the (first Chern Class $X(C_1) = 0$ where "X" is the proven Calabi-Yau manifold: one can say that taking the equation $< 1 + 3(\mathrm{JX} + \mathrm{Y}) > ^2$ with the minimal 1 and maximal 3 for the 3-fold of the Calabi-Yau as considered in String Theory: There's lies a relation to the Weil cohomology and Hodge - de Rham Spectral Sequence whose degeneration is crucial for the J-Holomorphic Polygons along with the Atiyah - Hirzebruch Spectral Sequence in $E_4^{p,q}$ sheets for the value of 4 where it's equal to Kähler manifold (the same when corresponds to Ricci flatness having the compact form then it's safe to say) = $Fukaya\ Category \equiv Morita\ Equivalence$ $(\text{for this } X(C_1)) = 0$. The nontrivial aspect to mention that in the construction of the hypercohomology when one finally arrives the Einstein − Kähler metric and proved the Calabi-Yau for the first Chern Class $X(C_1) = 0$ then for the X which when established as the Fukaya Category then the bimeromorphic relation to that (Calabi-Yau and Fukaya Category) is only true when the Kähler current (big and nef) is taken to prove that the Kähler is in Fujiki's Class C Manifolds. But the first construction must start with de Rham Complex (in Algebraic form) and then step-by-step taking this [derived category] with the degeneration leads to the first equivalence between de Rham Cohomology and Hodge - de Rham Spectral Sequence. There, the Hodge Diamond (W, V) can be easily seen in the Mirror Symmetrical Way through Homological Mirror SYMMETRY and dHYMT (deformed Hermitian Yang-Mills Theory) where we can use that [derived category] for $\boldsymbol{A}_\infty$ to prove the further correspondence between A and B Model. Detailed computations regarding this has been made in the paper.

SECTION I: CONSTRUCTING BIMEROMORPHIC EQUIVALENCE

I. **Fukaya Categories**

For a Riemannian manifold M where in the case of the first Chern class $c_1 \leq 0$ admits a Kähler – Einstein metric or not was first conjectured by Calabi and later proved by Yau. While for the case of $c_1 = 0$ and the vacuum EFE $T_{uv}dx^u dx^v = 0$; the Riemannian manifold M becomes the Calabi – Yau manifold such that $M = \rho$. This suffices a strong positivity condition for the Kähler potential σ such that in the Kähler – Einstein formalism, by looking at the metric $\omega_g + 2^{-i}\partial\bar{\partial}\sigma$; the condition lies as $\sigma = +ve$. Now, for the Kähler manifold in real (1,1) –form, for the exact differential ϵ_1 in the class $[\epsilon_1] \in H^{1,1}(\rho, \mathbb{R})$ one can define the potential to be $\sigma : \rho \longrightarrow \mathbb{R}$ such that $[\epsilon_{1_\sigma}] = [\epsilon_1]$ admits : $\epsilon_{1_\sigma} = \omega_g + 2^{-i}\partial\bar{\partial}\sigma$ gives the deformed Hermitian Yang – Mills equation (dHYM) for (ρ, ω) where the real part can be given as $Re(\omega_g + 2^{-i}\partial\bar{\partial}\sigma) > 0$ with $Im(\omega_g + 2^{-i}\partial\bar{\partial}\sigma) = 0$ implying the positivity condition for the Kähler potential σ such that in the Kähler – Einstein formalism in respect to the metric $\omega_g + 2^{-i}\partial\bar{\partial}\sigma$ where the dHYM provides a correspondence between the A – model and B – model of the equations of motion of D – Branes in string theory which can be seen in Hodge diamond for V in $A - model$ and W in $B - model$ termed as homological mirror symmetry [1-3].

For the complex manifolds having the holomorphic volume elements containing no such zeros; precisely the 3 –dim Calabi – Yau manifolds there exists a duality that in principle relates the Hodge numbers depending on the symmetry operations known as mirror symmetry; it has been stated that for the deg_n Calabi – Yau manifolds the correspondence is between the symplectic part ($A - model$) to the complex part ($B - model$) for $c_1 = 0$. The defined Hodge structure for the Kähler space ρ defined as for V and W the equations are defined as,

$$H^{p+q=k}(\rho, \mathbb{Z}) \otimes \mathbb{C} \cong \bigoplus_{p+q=k} H^q(\rho, \Omega^p)$$

$$\deg H^p(V, \Omega^p) = \deg H^{n-p}(W, \Omega^q)$$

Let there be a triangulated category $d(\mathcal{A})$ for which the $\mathcal{A}$ satisfies the Abelian to the category of A – modules where A is an associative algebra; the category $d(\mathcal{A})$ are the complex of free A – modules and the associated morphisms are in equivalence for the homotopy classes in differential graded morphism of deg_0. The derived bounded category $\bar{d}(\mathcal{A}) \overset{\subset}{\rightarrow} d(\mathcal{A})$ consisting of A – module complexes with nonvanishing cohomology groups. For the categories [4-7],

$$Hom_{d(\mathcal{A})}(M,N) := H^0\left(\bigoplus_{p+q=k}\prod_i Hom_{\mathcal{A}}\left(M^i, N^{i+k}\right)\right)$$

There are associated shift factors that shifts the degree of those complexes,

$$\begin{cases} M \to M[n], M[1]^{k+n} \\ (M[n])[m] = M[n+m], M[0] = M \end{cases}$$

This construction can be extended to the notions of twisted complex in the case of A_∞ –category $\mathcal{A}$ with a shift vector where for a one-sided twisted complex; there is a family of $\left(M^{(i)}\right)_{i\in\mathbb{Z}}$ for objects in A_∞ –category $\forall i\ \exists i > j$ there is a collection of morphisms $\tau_d^\Sigma\ \exists\Sigma \ni \gamma_{ij}$ for the shift,

$$Hom_{\mathcal{A}}\big(X[i], Y(j)\big) = Hom_{\mathcal{A}}(X,Y)[j-i]$$

$$\exists\forall\left(M^{(i)}\right)_{i\in\mathbb{Z}}$$

$$\text{for } \tau_d^\Sigma\big|_{\mathbb{Z}} \cong \bigoplus_{k,j} Hom_{\mathcal{A}}\left(M^{(j)}, N^{(j+k)}\right)[-k]$$

Let S be a closed symplectic manifold for $c_1 = 0$. For MS the space of pairs (y, M): where y is a point of M and S is a Lagrangian of subsets of T_yS for the space MS fibered over S for the fibres isomorphic to $\mathbb{Z}$. As, M and S are subsets; thus, for the nontrivial purpose denoting with the Lagrangian operator M as $\bar{\mathcal{L}}$ and S as $\mathcal{L}$ with the notion of a Floer cochain complex $\mathcal{A}_{\mathcal{F}^*}$ module generated by intersection points $\bar{\mathcal{L}} \cap \mathcal{L}$ can be viewed as a set of morphisms from $\bar{\mathcal{L}}$ to $\mathcal{L}$ sufficing the equation,

$$S\mathcal{A}_{\mathcal{F}^*}(\bar{\mathcal{L}}, \mathcal{L})$$

This A_∞ –category can have higher composition maps for twisted complexes of $\mathcal{A}$ as,

$$\bar{d}(\mathcal{A}) := H({}_{\tau_d^\Sigma}\mathcal{A})$$

Returning to the equation $1 + 3\langle PX + Y\rangle^2$ where for the almost complex structure P on the symplectic manifold S for the generators of $\tau_d^\Sigma\big|_{\mathbb{Z}}$; one can define $J-$ *holomorphic* polygons for $\prod_{d-1,d} \in S\mathcal{A}_{\mathcal{F}^*}(\bar{\mathcal{L}}_{d-1}, \mathcal{L}_d)$ and $\tilde{\prod}_{0,d} \in S\mathcal{A}_{\mathcal{F}^*}(\bar{\mathcal{L}}_0, \mathcal{L}_d)$ forms the equation,

$$\tau_d\left(\prod_{d-1,d}, \dots, \prod_{0,1,d}\right) = \sum_{S\mathcal{A}_{\mathcal{F}^*}(\bar{\mathcal{L}}_0, \mathcal{L}_d)\in\bar{\mathcal{L}}_0\cap\mathcal{L}_d} n\left(\left(\prod_{d-1,d}, \dots, \prod_{0,1,d}\right)\cdot \tilde{\prod}_{0,d}\right)$$

This sequence satisfies the A_∞ –categorical relations as the boundary of the $J-$ *holomorphic* polygons corresponding the configuration space of degenerate polygons. This A_∞ –category is the use of homological mirror symmetry where the mirror conjecture will apply for $(V,\omega) - dim_{2n}$ symplectic manifold for $c_1 = 0$ and W is its dual for dim_n complex algebraic manifold for the embedding of the Fukaya category

$\bar{d}(F(V))$ as a full triangulated category into $\bar{d}(Coh(W))$ for the isomorphism $(Hom(M,N))^* \simeq Hom(N,M[n])$ which for the A_∞ −category is cyclically symmetric. Moreover, this duality can be extended to string theory for $\mathcal{L}$ −varaities as the local boundary conditions for A – model while holomorphic vector bundles as the local boundary conditions for the B – model. Whereas the $J - holomorphic$ curve and the almost complex structure P in the equation $1 + 3\langle PX + Y\rangle^2$ is a trivial formalism for the parameter P and can be replaced by $1 + 3\langle JX + Y\rangle^2$ for the almost complex structure parameterization J.

2. Fujiki Class C Manifolds

Let us denote a compact Kähler manifold through a parameterization $(\bar{\rho}, b_k\omega_k^n)$ whose real closed forms are ϵ_1, ϵ_2 or α, β respectively is of $(1,1)$ −forms; where there exists a cohomology class that is nef. Then for the class $[\epsilon_1]$ or taken respectively here $[\alpha]$ is big iff $V_\alpha > 0$ where V is the volume. The Kähler current can be taken as $J \geq b\omega_k$ for some $b > 0$. Then there for $V_\alpha > 0$ exists a closed positive current by definition $\tilde{J}$ in the class $[\alpha]$ gives the equation for the manifold $(\bar{\rho}, b_k\omega_k^n)$; for the Kähler current $J \geq \int_{\bar{\rho}} J^n \geq b^n \int_{\bar{\rho}} \omega^n > 0$ is,

$$\int_{\bar{\rho}} \tilde{J}^n = \frac{\int_{\bar{\rho}} J^n \geq b^n \int_{\bar{\rho}} \omega^n > 0}{2} > 0$$

For the closed positive current $\tilde{J}_k \in [\alpha]$ there are analytic singularities with $J_k \geq -b_k\omega$ when $b_k = 0$ and with $J_k \longrightarrow \tilde{J}_k$ as $k \longrightarrow \infty$ then we get,

$$\lim_{k\to\infty} \inf \int_{\bar{\rho}} (J_k + b_k\omega)^n \geq \tilde{J}^n \; \forall k \text{ large;}$$

$$J_k \geq -b_k\omega$$

$$\int_{\bar{\rho}} (J_k + b_k\omega)^n \geq \bar{\Sigma} > 0$$

For $\bar{\Sigma} > 0$; a resolution of the singularities for J_k can be taken as $\tilde{\Sigma}_k : \bar{\rho}_k \longrightarrow \bar{\rho}$ $\exists \bar{\Sigma}_k$ is a composition of blows up of smooth centers for $\bar{\rho}_k$ proved to be a Kahler given in terms of $\tilde{\Sigma}_k J_k$ as $\tilde{\Sigma}_k^* J_k = -E_k \tilde{\Sigma}_k^* \omega + [\mathcal{E}_k]$ with $\bar{\Sigma}_k J_k$ comprises of $[\mathcal{E}_k]$ as the effective $\mathbb{R}$ −divisor and $-E_k \tilde{\Sigma}_k^* \omega$ is the closed real $(1,1)$ −form for the Bott – Chern class $[\alpha] \in H_{BC}^{1,1}(\bar{\rho}, \mathbb{R})$. Thus, we have the equation,

$$\int_{\bar{\rho}} (J_k + b_K\omega)^n = \int_{\bar{\rho}_k} \left(\tilde{\Sigma}_k^* J_k + b_K \tilde{\Sigma}_k^* \omega\right)^n \cong \int_{\bar{\rho}_k} \left(-E_k \tilde{\Sigma}_k^* \omega + \tilde{\Sigma}_k^* \omega\right)^n \geq \bar{\Sigma} > 0$$

Thus, the nef class $\left(-E_k \tilde{\Sigma}_k^* \omega + \tilde{\Sigma}_k^* \omega\right)^n$ on $\bar{\rho}_k$ shows $\beta = \tilde{\Sigma}_k^* \omega$ for a closed positive curvature J_k' on $\bar{\rho}_k$ gives,

$$J_k' \geq \frac{\int_{\bar{\rho}_k}\left(-E_k\tilde{\Sigma}_k^*\omega - \tilde{\Sigma}_k^* J_k\right)^n}{n\int_{\bar{\rho}_k}\left(-E_k\tilde{\Sigma}_k^*\omega - \tilde{\Sigma}_k^* J_k\right)^n \wedge \tilde{\Sigma}_k^*\omega}\tilde{\Sigma}_k^*\omega$$

Which is bounded and follows that there is a constant $\bar{\Sigma}' > 0\ \exists\forall k$ there is $J_k' \geq \bar{\Sigma}'\tilde{\Sigma}_k^*\omega$ for the value of k there is $(\tilde{\Sigma}_k) * (J_k' + [\mathcal{E}_k]) - b_k\omega \geq \omega 2^{-\bar{\Sigma}^{-1}} \in [\alpha]$ and is a Kähler current containing analytic singularities and any non – Kähler locus $\mathcal{E}_\alpha$ is intersection of all loci in $[\alpha]$; then $\mathcal{E}_\alpha \overset{\subset}{\rightarrow} \bar{\rho} \forall\ [\alpha] \neq big;\ \mathcal{E}_\alpha \equiv \bar{\rho}$. Thus, for the cohomology class nef and big in $[\alpha]$ for $(1,1)$ –form real closed; one gets a union of $\bar{N} \overset{\subset}{\rightarrow} \bar{\rho}$ for the equation with the analytic singularities,

$$\mathcal{E}_\alpha = \bigcup_{\int_{\bar{N}} \alpha^{dim_{\bar{N}}} = 0} \bar{N}$$

Now, for the null locus of the set on the class $[\alpha]$; the existence of a big class on $\bar{\rho} \Longrightarrow \bar{\rho}$ is in Fujiki's class C manifold $\xrightarrow{bimeromorphic}$ compact Kähler manifold [8-11].

SECTION II: CONSTRUCTING MORITA AND FUJIKI'S CLASS C MANIFOLD EQUIVALENCE

I. Morita Equivalence for Hilbert C^*-Module

Any map from a domain to a codomain with the mapping parameter $\theta : \zeta \longrightarrow \zeta'$ can provide a continuous set of functions when ζ and ζ' is endowed with a metric which when attempt for any representation of a Topological structure considering two sets $\{\zeta\}\ and\ \{\zeta'\}$ there norms even a bijection[1] between them $\zeta \ \leftrightarrow\ \zeta'$ which for a defined function f over a value of $f(x)$ there involves a structure of a vector space with concerned operations through a continuous linear transformation, that space for that function carries a Topology best known as Hilbert space. The specified module that carries the $c* -$ algebra for that space is defined as $c*-$ Hilbert modules[3] through the inner product.

For any group ^ with a subgroup ℓ the representations $\Gamma_\ell^\wedge$ makes it easier to construct new representations through the subgroup or the smaller group ℓ over certain parameters that when categorize through the constructive modules of Hilbert's $c*$ then this extent the $c* -$ module to $c* -$ algebras through the non–commutative formulations.

Furthermore, any derived pathway to construct the noncommutative geometry provides a framework for the moulder category to represent an equivalence over $(left - right) - symmetric$ rings as established afterwards with rings R and R'; then for the $ring - representations$, studying the category of those modules; there exists Morita equivalence for the isomorphic commutative form or in general norms in the case of $non - commutative$ rings.

For the constructions of $KK - Theory$; Morita equivalence is an important tool to c *− algebras where for the inequality on the two modules $A\ and\ B$; for the moulder form E on A and B for the moulder form E on A and $E\ \cdot$ on B (*as appeared later in the paper*) a homotopy invariant bifunctor can make a Morita equivalence for the $KK - Theory$ through $KK(A,B)\ and\ KK(B,C)$ for $A,\ B,\ C$ as c *− algebras; there's for the modular form E having elements ε, ϵ the inequality represents the form $< \varepsilon, \epsilon >< \epsilon, \varepsilon > \leq || < \varepsilon, \varepsilon >\ || < \epsilon, \epsilon >$ where for the $A - module$; the above relation holds and taking the $B - module$ representing the c *− algebraic pair $KK(A,B)\ and\ KK(B,C)$ where one finds the combined form over the composition product representing $KK(A,C)$ and the Morita equivalence to be represented in a specific formulation as to be proved throughout the paper[12,13].

Over the compact Hausdorff spaces[14] and considering the Fredholm modules of Atiyah–Singer Index Theorem[15] for a relatable definition of A, B, C in c *− algebras the Kasparov's product $KK(A,C)$ for $KK(A,B)\ and\ KK(B,C)$ will be established over an elliptic differential operator $\varrho^0_{M_s} or\ \varrho^0_{M_n}$ for $s - smoothness\ or\ n - dim$ and through extensive analysis of that operator which indeed suffice the Fredholm module making a relatable framework for $K - Homology$ and $K - Theory$ [16,3,5,6]; The Thom isomorphism is established for the Chern Character Ch over a mapping parameter ι through a $rank - n$ vector bundle $v_{1(n)}$ with v_2 having the first related to a unit sphere bundle. This in turn induces the categorical correspondence between a relational establishment over noncommutative geometry and noncommutative topology taking the function f over a bounded structure through linear transformations that bounds the concerned subsets $I\ and\ J$ for a mapping parameter ρ_η in the same Hilbert space H.

This will deduce for a much more concrete formalism of the $K - Theory\ to\ K - Homology$ with an extension of c *− algebras to $reduced\ c^* - a\lg e\ bras$ for parent group (^) that defined the ℓ^2 norm of Hilbert space taking into consideration the $KK - Theory$ with Gromov's $a - T - menable$ property for all the necessary formulations concerned before except Morita equivalence that when established through $5 - parameters$ through an assembly mapping parameter ℸ over discrete torsions gives the ultimate relation of $KK - Theory$ in $Baum - Connes\ conjecture$ taking into account both the $Novikov\ conjecture$ and $Kadison - Kaplansky\ conjecture$ for injectivity and surjectivity respectively connecting to noncommutative topology.

- Extensions have been made in the operator and Topological aspects in the cohomology class where several classifiers are shown with distinct property to suffice the Sp_c – Structure and the Atiyah – Hirzebruch spectral sequence for the Type II (II-A and II-B) as concerned on the complex Topology space T* where the Atiyah – Singer Index Theorem taking the Fredholm modules as necessary for K-Theory with Bott – Periodicity is taken and a channelization is made to Grothendieck – Riemann – Roch; for the transition of KK-Theory to Strings; Hodge dual, Gauge symmetry, charge density for the required Lagrangian in RR-fields through D-Brane Potential, De Rham Cohomology, and GSO – Projections are shown. P-form electrodynamics with P-Skeleton are considered for the purpose. NS 3-form and its relation to RR-flux in both D-Brane charge density and supergravity is established. The spectral sequence of Atiyah-Hirzebruch is taken and operator over $E_n^{p,q}$ for n taking the values 2,3,∞ over a consideration of several orders of K-Theory as such Topological, Algebraic, and Twisted. $\acute{e}$tale cohomology and its representation is shown for Algebraic K-Theory and the K$\ddot{a}$hler (without any specific consideration of compact and Ricci flatness) has been shown in general terms for K-Theory in a Twisted formalism in E_i for $i = 4 = \infty$.

For a Hilbert space H with a c*-module H_c one can define a c*-algebra for the metric g on a Riemann manifold M ($having\ the\ form\ M_g$) with a vector bundle V there exists a compact neighbourhood being locally variant on a small patch; over an isomorphism of the Hilbert space of that vector bundle V in a continuous way for a commutative c*-algebra through the vanishing infinity.

For the modular form of c*-algebra the Hilbert module for the non−commutative form is the generalized norm taking the algebra over a topological field T in unital formulation for the unit parameter $\imath$ as such for every ϵ in the algebra there exists $\epsilon = \imath\epsilon = \epsilon\imath$.

Representing over the induced form for any finite group ^ with $\ell \subset$ ^ for the vector bundle $\mathcal{V}$ on the Hilbert space H, any construction can be defined over the k −elements of the group ^ over L defined a parameter $\mathcal{P}$ as,

$$\mathcal{P} = \sum_{k=1}^{n} L_k$$

This gives for each k, the induced representation through group ^ in the same $L_k^+ \in L_k$ for $\ell \subset$ ^ through the vector representation $\mathcal{V}$ of subgroup ℓ being $\ell \subset$ ^ in Hilbert space H parametrized through,

$$\mathcal{X}_{(\pi,\mathcal{V})}$$

Thus, one gets,

$$\text{for every } \bigoplus_{k=1}^{n} L_k^+\mathcal{V} \text{ there is,}$$

$$\sum_{k=1}^{n} L_{(1,\ldots\ldots,n)k}\,\pi(L_k^+)\mathcal{E}_k$$

Representing $\mathcal{E}_k \in \mathcal{V}$, three non-trivial actions can be noted for the constructions,

1. $\mathcal{E}_k \in \mathcal{V}$
2. $L_k^+ \in L_k \forall \ell \subset$ ^
3. $\ell \subset$ ^

This takes a pre-Hilbert Hausdorff space to construct c*-algebra satisfying the operations of an inner product through the Hilbert A–module being non–negative and self-adjoint. Taking the inner product of the complex manifold representing $\mathcal{M}^*$ through,

$$\mathcal{M}^* \times \mathcal{M}^* \longrightarrow A$$

Thus, for any sequence of set that is countable over the Topological space T with a proper representation for the previously encountered manifolds $\mathcal{M}^T$ taking k^{th} countable order of infinity,

$$\{\mathcal{M}_k^T\}_{k=1}^{\infty}$$

When merged with the unital form taken before $\epsilon = \imath\epsilon = \epsilon\imath$ such that for every unit parameter $\imath$ there exists ϵ in the algebra; where for any c*-algebra there holds the Banach–algebra for a compact $\mathcal{F}$, that if provided there exists three forms taking $B_0(\mathcal{F})$[3,19,20,22],

1. $Typical\ form -$ For the complex space $\mathcal{M}^*$; the locally compact Hausdorff space for vanishing infinity norm gives $B_0(\mathcal{F})$ for continuous functions on $\mathcal{M}^*$.
2. $Unital - \begin{cases} \text{if is commutative} \\ \text{identity element of having norm 1} \\ \mathcal{F} \text{ in } B_0(\mathcal{F}) \text{ is compact} \end{cases}$
3. For Point [2] to have a congruent transformation, there is Banach algebra $B_0(\mathcal{F})$ in A–form where the congruent transformation is unital for a closet set $[A]$.

For the compact Hausdorff (here parameterizing $\mathcal{F}_0^+$) with vector bundles $\mathcal{V}$ for the labeling of $\mathcal{F}_0^+$ - 0 for positive to extend over Bott Periodicity with + as adjoint through 8–periodic homotopy groups from π_0 to π_7 such that[12],

$$\pi_{0,1,2,3,4,5,6,7} \text{ gives } 3 - \text{category tables in unitary } U, \text{orthogonal } O, \text{symplectic } Sp,$$

$$\begin{array}{ccc} \underline{U} & \underline{O} & & \underline{Sp} \\ \pi_k \longrightarrow \pi_{k+2} & \pi_{k+8} & \xrightarrow{=} & \pi_{k+4} & \forall k = 0,1 \ldots \\ & \pi_{k+4} & \xleftarrow{=} & \pi_{k+8} \end{array}$$

Thus, for Hausdorff $\mathcal{F}$; the underlying K-Theory $K(\mathcal{F})$ there is[12,23];

I. Topological K-Theory $\Rightarrow on\ \mathcal{M}^T\ for\ K(\mathcal{F})$
II. Reduced K-Theory $\Rightarrow K_{red}(\mathcal{F})\ for\ S^n \exists n > 0$ relates the Bott for $positive\ 0\ for\ \mathcal{F}_0^+$ and $adjoint +$ in Hausdorff $\mathcal{F}$ for $K_{red}(\mathcal{F}_0^+)$ in non–commutive form.

Where Point [I] relates the Banach–algebras for the locally compact Hausdorff over a abelian module on any sequence of set countable over Topological space T (as previously mentioned) on c*-algebras for bivariant forms suffice the proper framework for the Hilbert c*-module on rings R and R' for modular homeomorphisms on R such that the biproduct exists in finitary over a defined functor δ preserving equivalence and additive properties,

$$\delta : mod - R \longrightarrow mod - R'$$

$$\delta' : mod - R' \longrightarrow mod - R$$

$$\begin{cases} \delta : mod - R & \longrightarrow & mod - R' \\ \delta' : mod - R' & \longrightarrow & mod - R \\ & & \begin{cases} suffice\ Morita\ Equivalence\ (strong) \\ for * -operations\ on\ c * -algebras \end{cases} |_{R^{Morita} \approx R'^{Morita}} \end{cases}$$

For the naturally induced isomorphism for functors δ and δ' for a finite module ring R for the bi–module (R, R') suffice the natural isomorphism iff for $X_{(R,R')}$ and $Y_{(R',R)}$ there is,

$$(R, R') \quad - \ bimodule \ \Rightarrow \quad X_{(R,R')} \otimes_{R'} Y_{(R',R)} \cong R$$
$$(R', R) \quad - \ bimodule \ \Rightarrow \quad Y_{(R',R)} \otimes_{R} X_{(R,R')} \cong R'$$

Moreover, if we consider $A,\ B\ and\ C$ as c*-algebras then if there is a Hilbert B -module that is fully countably generated in the form of E , then for that c*-subalgebras of B there exists a strong Morita equivalence between A and B provided for the B module there is $\varphi\ (E) \cong\ A$ and for A module there is $\varphi\ (E\ \cdot\ )\ \cong\ B$ where for the c*-algebraic pair (A, B), over a homotopy invariant bifunctor the constructions can be taken for A , B and C in such a way that for the defined abelian group $KK(A, B)$ and combining it with $KK(B, C)$ a strong Morita equivalence can be established in the form,

$KK(A, B)\ \cong\ KK(A, C)\ \exists$ Combining the elements of $KK(A, B)$ AND $KK(B, C)$, there exists the product and the non-trivial assumptions that B and C are strongly Morita equivalent.

2. Type II Strings relation with Calabi-Yau through Twisted K-Theory

The $K - Theory$ for the operator and Topological aspects in the cohomology class; there exists distinct classifiers for the $D - Branes$ or Dirichlet Branes in the Ramond–Ramond (RR)– Sector of Type II-B Strings sufficing the $3 - dim$ integral class property. There is the cohomology class for the transformation–twist giving the $mod - 2$ torsion quantum corrections considering the Freed–Witten discrepancies as and when considered in the peculiar $K - Theory$ in the reconciled aspects over Atiyah–Hirzebruch spectral sequence.

The non–trivial aspect to discuss in high energy physics for the Topological K–Theory taking the Type–II (II-A and II-B) superstrings is to consider the RR–fields in $P - form$ electrodynamics considering the $10 - dim\ Supergravity$ for the potential $\mho^{\circ}$ over Ω_{P+1} $- field$ defined through the Hodge duals $*_d$ in the form Ω^{*d}_{9-P} there exists $4 - classifiers$ that will ultimately result the approach of $K - Theory$ in the complex Topological space T^* on manifold M over a representation M^*_Trelates not only the Atiyah–Singer Index Theorem (for the Fredholm modules, Bott–Periodicity as taken earlier) but also gives the Grothendieck–Riemann–Roch Theorem on bounded complex Λ^* on sheaves $S_{,,}$ over a relation $S_{,,}^{\Lambda^*}$ taking the morphism $\sigma_m : X \longrightarrow Y$ for $\sigma_m : A(X) \longrightarrow A(Y)$ over the Tangent sheaf T_{Λ^*} of Λ^* on $\sigma_m!$ to suffice $ch(\sigma_m! \Lambda^*)$ gives,

$$\Lambda^*_{\sigma_m}\left(ch(S_{,,})Td(T_{\sigma_m})\right)$$

All suffice through the $4 - classifiers$ as mentioned above[27,28],

1. Hodge dual $*_d$
2. Gauge symmetry $\mathring{g}_{P-form}$
3. Equations of motion $\partial * g^{\circ} = *\mathbf{J}$ for $\mathbf{J}_{\text{P-vector}}$
4. Charge density C_ρ through the Lagrangian for ζ_{C_ρ} in $RR - fields$ for ϖ_{10-P} through the D–Brane potential $(10 - P)$ gives the equations of motion $S^{\times}$ for $(10 - P)$having a replacement order of $P\, to\, (7 - P)$ for the previously taken charge density C_ρ giving two non–trivial relations[29,30],

   A. $De\ Rham\ Co\hom o \log y$ with $H - twist$ for the exterior derivative ∂ with charge density C_ρ for the parameter χ gives,

   $$\partial\chi_{9-P} + H \times \Omega_{9-P}$$
   $$= \partial\chi_{P+1}$$
   $$= \partial^2\varpi_{7-P}$$
   $$= C_{9-P}$$

   B. The action for Type II (II–B being both T and S–dual to itself) for non–invariant GSO – projections in subdomains where for the existence of 32–supercharges in Type II–B $(\mathcal{R}^{8,1} \times S^1)$ the action $S_{,,}$ of P–form electrodynamics on a manifold M through gauge symmetry can be represented by $\mathring{g}_{P-form}$ gives,

   $$S_{,,} = \int_M \left[\frac{1}{2} g^{\circ}\chi * g^{\circ} + (-1)^P B\chi * J\right]$$

   Which gives the nilpotent potential in manifold M over a spacetime coordinates (σ, τ) as,

   $$\partial\Omega_{P+1} + \chi_{9-P} + \Omega_{\ (\sigma,\tau)}$$
   $$= \partial\Omega_{P+1\ (\sigma,\tau)}$$
   $$= \partial^2\,\varpi_{P\ (\sigma,\tau)}$$
   $$= 0_{\ (\sigma,\tau)}$$

All of these suffice for Sp_c in the extension of Poincare duality in a generalized norm of orientability of homology theory taking the Thom Isomorphism in complex form of Topological $K - Theory$ relating Atiyah–Singer Index Theorem and Fredholm modules, Bott–Periodicity, Atiyah–Hirzebruch, Grothendieck–Riemann–Roch with KK–Theory[31-34].

Additionally, to discuss furthermore about the Type II Superstrings formalism as associated with supergravity for a homology class there is a relation between the Dirac quantization conditions and RR–fields where in the Lie group structure,

$$U(1) \times SU(2) \times SU(3) \subseteq SU(5) \subseteq SO(10) \subseteq E(8)$$

The Photon being represented by $U(1)$ the related methodology of the charge quantization and the magnetic monopoles where their independent nature relates the breaking of gauge group from $D(1)$heavy branes when the distance is infinite for a path υ suffice the relation,

$$\prod_{\upsilon}\left(1 + ieA_j \frac{dx^j}{d(\upsilon)} d(\upsilon)\right) = \exp\left(ie \int A \cdot d(\upsilon)\right)$$

$$\exists\, e \oint_{\partial \cap} A \cdot d(\upsilon) = \int_{\cap} Bd(\upsilon)$$

Considering a cycle σ_{cy} in the homogeneous Lie group, the movement can ultimately results in lifting the Lie group that originates over identity structures through,

$$2 - times(\sigma_{cy})\ and\ 3 - times(\sigma_{cy})$$

Where the $2 - times(\sigma_{cy})$ where a covering parameter $\Im$ for $SO(2)$ can maintain the Type II superstring actions over the $Twisted\ K - Theory\ (over\ Topo\log i\ cal\ norms)$.

One category of Type II superstrings (Type II-B) which has been extended to $12 -$ dim where in the t'Hooft limit, for Yang–Mills $N = 4$, F-Theory being encountered under $SL(2, \mathbb{Z})$, the D–Brane analogy being extended where there exists some non–trivial aspects being existent over RR–Fields and its relation to the Twisted K–Theory making up these points,

1. $GSO -$ Projections for an eliminated Tachyon and preserved Supersymmetry.
2. Distinct classifiers for Type II into II_{II-B}^{II-A}.
3. $SL(2, \mathbb{Z})$ for a CFT for a worldsheet periodicity as concerned for Fermion–projections giving 3 sub–relations,
    a. Invariance over $SL(2, \mathbb{Z})$.
    b. Modular diffeomorphisms as expressed on Torus for Point [3] to get rid of gravitational anomalies.
        i. This in turn establishes the integral for $Kalb - Ramond\ (K - R)\ field$ with the relation to the **B** field for ℷ as,

$$- \int_{KR} ℷ^i\, ℷ^j\, \boldsymbol{B}_{ij}$$

Thus, for the correspondence to $KR\ NS - NS\ B - field$; a far more concrete relation can be attained for $H - flux_{D-Brane}^{NS}$where the $P - form\ for\ P -$

$skeleton$ represents a complicated structure later but for the cohomology integral coefficients for a D-Brane absent RR–flux the relation can be stated over,

$$NS\ 3-form^{\otimes RR-flux \cong ch \arg e\ density\ of\ D-Brane}_{+RR-flux \cong equations\ of\ motion\ (\sup ergravity)}$$

Extending Type II for Type II–B the representation when made for a manifold M for the group operators Og in the quotient space $q\ with\ q^{\partial-rescalling}$ Type II–B represents the Orientifold over the operator relation where $\partial\ in\ \partial - rescalling$ being taken trivially for the involution parameter, the non–empty operator represents the orientifold for the operator Og_p^2 such that for the operator $P\sim$ there is Type II–B for,

$$\partial(P\sim)$$

Where through the splitting another structure represents $II-A$ for the $(1-1)-form$.

The $P-skeleton$ as stated above in turn gives the Topological K–Theory over **B** for the fibre $f.$in the cohomological space $M.$ over a Serre fibration parameter $S_{f.}\colon M. \to B.$ in the $(p,q)-norm$ representing the cohomolgy pair $\left(M._{(p)}, M._{(p,q)}\right)$ for $k^{th}-co\hom o\log y\ \ group$ through,

$$\bigotimes_{p,q} H^k\left(M._{(p)}\right)$$

$$\bigotimes_{p,q} H^k\left(M._{(P)}, M._{(P),}M._{(P-1)}\right)$$

For the Atiyah–Hirzebruch taking the space $M.$ and the spectral sequence associated with it for the fibres $f.$ there exists the $E_n-sheet$ taking $(p,q)-norms$ for $E_n^{p,q}$ for n taking the values $2,3,\infty$; the spectral sequence can be in respect of the differentials E_d where there is,

1. Atiyah–Hirzebruch spectral sequence
2. Twisted K–Theory
3. Topological K–Theory
4. Algebraic K–Theory
5. Complex δ
6. E_n for different values of n providing;
    a. Serre spectral sequence for E_1
    b. Topological K–Theory for E_2^p
    c. Twisted K–Theory for E_3 over the differential $E_{3_{(d)}}$ such that for the $E_n^{p,q}$; n takes an equality for $E_2\ and\ E_3$.
    d. For [Point b] in complex parameter $\delta. = 2k+1$denoting complex projective $\mathbb{C}\mathrm{P}^{\delta}$ there exists two foundations,
        i. Collapsing for $even\ 2k$
        ii. Non collapsing for $odd\ 2k+1$

1. Where Topological K–Theory as associated with Atiyah–Hirzebruch for $2k+1$ over space $M.$; a nice relation can be expressed in $E_2^{p,\delta.}(M.)$.

7. For the K$\ddot{a}$hler where any compact Kahler having Ricci flatness is a Calabi–Yau for all the threefold being non–trivial in superstring theory, any K$\ddot{a}$hler (*without any consideration of being compact*) can give the twisted formalism of K–Theory for E_i such that $i \equiv 4 = \infty$.
8. Algebraic K–Theory having a relation to the $\acute{e}tale$ cohomology for the scheme M_T^e where $M_T^{e\prime}$ is a Topological space; any representation can be done in the local isomorphism such that for the category taking M_T^e for $etale\ representation\ et(M_T^e\ )$suffice isomorphism for the Topological space $T(or\ M_T\ )$ which provides the Atiyah–Hirzebruch spectral sequence for E_2 in $(p,q) - norms$ thereby establishing the Quillen– Lichtenbaum conjecture for $E_2^{p,q}$ with the $\dot{e}tale$ cohomology M_T^e [2,3,5].

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