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COVID-19 Epidemic Model Including Unreported Cases and Stochastic Noise

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Abstract

The study of infectious diseases in humans has become increasingly important in public health. This paper extends the SEIR model to include unreported COVID-19 cases (U) and environmental white noise. Dynamic analysis is conducted based on the variation of the environment. The ergodicity and stationary distribution criteria are discussed. Using a Lyapunov function, we write down some sufficient conditions for disease extinction. With different intensities of stochastic noises, we calculate the threshold of extinction for the stochastic epidemic system. In order to control the spread of disease, the stochastic noise plays an important role. A numerical simulation and a fit to real data have shown that the model and theoretical results are valid.

Keyword: COVID-19; Stochastic perturbations; Extinction; SEIR model; Curve-fitting

1 Introduction

In order to prevent the spread of this infectious disease in communities, immediate actions, as well as intensive research, are necessary [1, 2]. On December 8, 2019, several Coronavirus cases were reported in Wuhan, China. It was previously unknown that Coronaviruses could infect humans. Even so, it has been found to cause widespread autoimmune reactions in some patients for unknown reasons [3]. In addition to causing human-to-human infections, the virus eventually caused a global pandemic. According to epidemiological studies [4], coughing or sneezing is the main route of transmission for COVID-19.

Due to the fact that this pandemic was a new disease, its spread and mortality were unknown, it also attracted the attention of researchers from a variety of fields. In order to figure out the transmission dynamics of the disease and predict its development among different populations, epidemiologists, statisticians, and mathematicians developed models. Infectious disease modeling plays a crucial role in disease control. Literature has described and investigated a variety of epidemic models. In [5] the authors considered the dynamics of a SIR-based COVID-19 model with linear incidence rates, nonlinear removal rates, and public awareness. Ziren *et al.* [6] presented an analysis of the SEIR model for spreading COVID-19 based on the unreported infected population and dynamic parameters. Authors of [7] investigated SEIR epidemic models with vaccination, time delays, and stochastic perturbations by including white noise in some parameters.

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Furthermore, the transmission of COVID-19 is disturbed by various random factors in the environment. A number of environmental factors influence the spread of the new strain of Coronavirus COVID-19 to humans, including humidity, precipitation, temperature, and awareness of people. An increasing number of researches consider environmental noise and study stochastic models for COVID-19 [8]. In fact, environment disturbances have an important effect on the evolution of infectious diseases [9, 10], and Gaussian white noise is usually selected as an appropriate representation of environmental fluctuations. There are different possible approaches to include random effects in the model, both from a biological and from a mathematical perspective. For instance, Liu et al. [11] and Luo et al. [12] proved that a large disturbance of white noise can lead infectious diseases to extinction. A large number of works indicate that stochastic disturbance can suppress disease outbreak [13].

This study extends the analysis to the Susceptible-Exposed-Infectious-Unreported-Removed (SEIUR) model for COVID-19 infectious diseases with stochastic perturbations and discusses its qualitative behavior. The model is verified by curvetting or real observations in UAE, using least-squares approach. In Section 2, we provide a stochastic mathematical model for COVID-19 governed by the SEIUR model. In Section 3, we investigate the stochastic analysis for the model with its qualitative behavior. The final section provides some numerical simulations and concluding remarks.

2 The Model

Despite being applicable to most infectious diseases, SEIR has some shortcomings when applied to COVID-19 data. COVID-19 can also be transmitted by those who are exposed to it. During incubation, SEIR assumes that members of the E compartment are infected but not infectious, so-called unreported cases. This paper proposes a piecewise SEIUR model; Figure 1 illustrates it. First, the model assumes that susceptible individuals are born at the rate $\eta(S;E;I;U;R)$; this rate is a function of the densities of susceptible, exposed, infected, unreported cases, and recovered individuals. The proportion of exposure to infected is $f \in (0,1)$, and the proportion of exposed to unreported cases is $(1-f)$. The model is as follows:

$$\begin{aligned}
\frac{dS(t)}{dt} &= \eta - \beta S(t)E(t) - \mu S(t)\\
\frac{dE(t)}{dt} &= \beta S(t)E(t) - (\sigma+\mu)E(t)\\
\frac{dI(t)}{dt} &= \sigma f E(t) - (\gamma+\mu)I(t)\\
\frac{dU(t)}{dt} &= \sigma(1-f)E(t) - (\gamma+\mu)U(t)\\
\frac{dR(t)}{dt} &= \gamma(I(t)+U(t)) - \mu R(t).
\end{aligned} \tag{1}$$

The parameters in (1) are summarized in Table 1. We explore the qualitative behavior of system (1) by studying the local stability of the disease-free equilibrium (DFE) and the endemic equilibrium (EE). The DFE is $\mathcal{E}_0 = (S_0, E_0, I_0, U_0, R_0) = (\frac{\eta}{\mu}, 0, 0, 0, 0)$. However, the EE $\mathcal{E}^* = (S^*, E^*, I^*, U^*, R^*)$, where $S^* = \dfrac{\sigma+\mu}{\beta}$, $E^* = \dfrac{\mu}{\beta}[\mathcal{R}_0 - 1]$, $I^* = \dfrac{\sigma\mu f(\mathcal{R}_0-1)}{\beta(\gamma+\mu)}$, $U^* = \dfrac{\sigma\mu(1-f)(\mathcal{R}_0-1)}{\beta(\gamma+\mu)}$, $R^* = \dfrac{\gamma\sigma(\mathcal{R}_0-1)}{\beta(\gamma+\mu)}$. By using the next-generation matrix (NGM) [18], we

Table 1: Description of model (1) parameters.

Parameters	Description
η	rate of newborn in the population
β	contact/transmission rate
μ	mortality rate
σ	incubation period
f	the proportion of exposure to infected
γ	the recovery rate

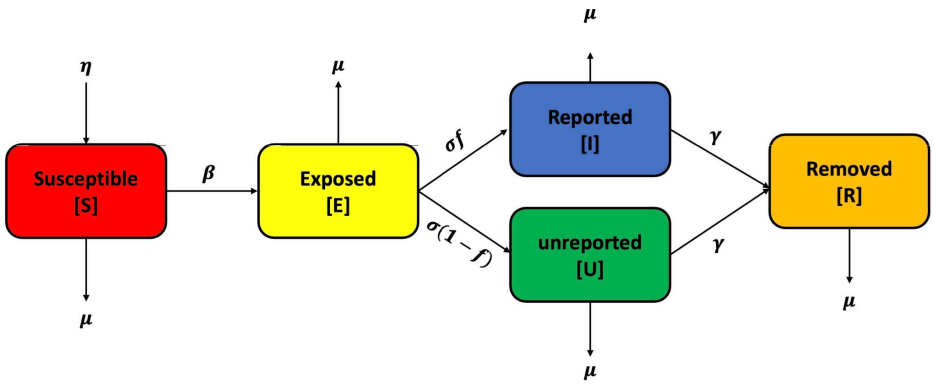

Figure 1: Scheme of SEIUR model (1).

calculate the basic reproduction number $\mathcal{R}_0 = \dfrac{\beta S_0}{\sigma + \mu} = \dfrac{\beta \eta}{\mu(\sigma + \mu)}$.

The transmission of diseases is disturbed by numerous random factors in the environment. Taking this into thoughts empowers to show uncertainty into deterministic natural models to uncover the natural inconstancy impact, whether it could be an environmental perturbation in parameter or random noise within the differential systems. Therefore, we assume that the stochastic perturbations are of the white noise type which are directly proportional to $S(t)$, $E(t)$, $I(t)$, $U(t)$ and $R(t)$ in model (1) respectively [15, 16]. Thus, we propose a stochastic model of the following form

$$\begin{aligned} dS(t) &= [\eta - \beta S(t)E(t) - \mu S(t)]dt + \psi_1 S dW_1 \\ dE(t) &= [\beta S(t)E(t) - (\sigma + \mu)E(t)]dt + \psi_2 E dW_2 \\ dI(t) &= [\sigma f E(t) - (\gamma + \mu)I(t)]dt + \psi_3 I dW_3 \\ dU(t) &= [\sigma(1-f)E(t) - (\gamma + \mu)U(t)]dt + \psi_4 U dW_4 \\ dR(t) &= [\gamma(I(t) + U(t)) - \mu R(t)]dt + \psi_5 R dW_5. \end{aligned} \tag{2}$$

Subject to the following initial conditions:

$$S(\chi) = \zeta_1(\chi), \quad E(\chi) = \zeta_2(\chi), \quad I(\chi) = \zeta_3(\chi), \quad U(\chi) = \zeta_4(\chi), \quad R(\chi) = \zeta_5(\chi). \tag{3}$$

$\zeta_i(0) > 0$ and $\zeta_i(\chi)$, $i = 1, \ldots, 5$, are nonnegative continuous initial functions, where ψ_i, $i = 1, \ldots, 5$ are the intensities of white noise. $W_i(t)$, $i = 1, \ldots, 5$ stand for the independent Brownian motions defined on a complete probability space $(\Omega, \mathcal{M}, \{\mathcal{M}\}_{t\geq 0}, P)$ with a filtration $\{\mathcal{M}_t\}_{t\geq 0}$ satisfying the usual conditions.

3 Main Results

Theorem 1. *For any initial value* $(S(0), E(0), I(0), U(0), R(0)) \in \mathbb{R}^5_+$*, there is a unique positive solution* $(S(t), E(t), I(t), U(t), R(t))$ *for system (2) on* $t \geq 0$ *and the solution will remain in* $\mathbb{R}^5_+$ *with probability one.*

Proof. Since the system coefficients (2) satisfy linear growth and Lipschitzian conditions we can show that system (2) has a global positive solution. Therefore, we need to establish a Lyapunov function, so we define

$$\begin{aligned}\mathcal{K}(S,E,I,U,R) = \mathcal{K}(.) = (S-1-\ln S) + (E-1-\ln E) + (I-1-\ln I) + (U-1-\ln U) \\ + (R-1-\ln R).\end{aligned} \tag{4}$$

By Itô's formula on $\mathcal{K}$

$$\begin{aligned}d\mathcal{K}(.) = \mathcal{L}\mathcal{K}dt + \psi_1(S-1)dW_1(t) + \psi_2(E-1)dW_2(t) + \psi_3(I-1)dW_3(t) \\ + \psi_4(U-1)dW_4(t) + \psi_5(R-1)dW_5(t).\end{aligned} \tag{5}$$

Such that

$$\begin{aligned}\mathcal{L}\mathcal{K} &= \eta - \mu S - \frac{\eta}{S} + \beta E - \mu E - \beta S - \mu I - \frac{\sigma f E}{I} - \mu U - \sigma(1-f) - \mu R - \frac{\gamma I}{R} \\ &\quad - \frac{\gamma U}{R} + 5\mu + \sigma + 2\gamma + \frac{\psi_1^2 + \psi_2^2 + \psi_3^2 + \psi_4^2 + \psi_5^2}{2} \\ &\leq \eta + 5\mu + \sigma + 2\gamma + \beta(E-S) - \mu(S+E+I+U+R) + \frac{\psi_1^2 + \psi_2^2 + \psi_3^2 + \psi_4^2 + \psi_5^2}{2} \\ &\leq \mathcal{H}.\end{aligned} \tag{6}$$

Such that $\mathcal{H}$ is a positive constant. Hence, $\mathcal{L}\mathcal{K}$ is bounded. The rest of the proof is standard [7], so it is omitted. □

In the next theorem, we focus on the existence of stationary distribution of model (2). From biological point of view, stationary distribution can be assumed as a weak stability of the system, in which the infection is persistent in the time mean sense. Let $V(t)$ is a regular time-homogenous Markov process in $\mathbb{R}^d$, defined by the stochastic differential equation

$$dV(t) = g(V(t))dt + \sum_{s=1}^{d} f_s(V(t))dW_s(t). \tag{7}$$

Such that, the diffusion matrix of the process $V(t)$ is

$$\Pi(v) = (\varsigma_{ij}(v)), \quad \varsigma_{ij}(v) = \sum_{s=1}^{d} f_s^i(y) f_s^j(v).$$

Lemma 1. *[14]. The Markov process* $V(t)$ *has a unique ergodic stationary distribution* $\pi(.)$ *if there exist a bounded domain* $\mathcal{N} \subset \mathbb{R}^d$ *with regular boundary* Δ *and*

C1: there is a positive number $\mathcal{M}$ *such that* $\sum_{i,j=1}^{d} \varsigma_{ij}(v)\xi_i\xi_j \geq \mathcal{M}|\xi|^2, v \in \mathcal{N}, \xi \in \mathbb{R}^d$.

C2: there exists a nonnegative $\mathcal{C}^2$*-function* $\tilde{D}$ *such that* $\mathcal{L}\tilde{D}$ *is negative for any* $\mathbb{R}^d \setminus \mathcal{N}$.

Assume that

$$\mathcal{R}_0^s = \frac{\eta\beta\gamma\sigma^2(1-f)}{\hat{\sigma_1}\hat{\sigma_2}\hat{\sigma_3}\hat{\sigma_4}\hat{\sigma_5}}. \tag{8}$$

Where $\hat{\sigma_1} = \sigma + \mu + \frac{\psi_2^2}{2}$; $\hat{\sigma_2} = \gamma + \mu + \frac{\psi_3^2}{2}$; $\hat{\sigma_3} = \gamma + \mu + \frac{\psi_4^2}{2}$; $\hat{\sigma_4} = \mu + \frac{\psi_5^2}{2}$ and $\hat{\sigma_5} = \mu + \frac{\psi_1^2}{2}$.

Theorem 2. *If $\mathcal{R}_0^s > 1$, then there exists a unique stationary distribution for system (2) and it has the ergodic property.*

Proof. To verify condition (C.1) of Lemma 1, the diffusion matrix of system (2) is given by

$$\mathcal{Z} = \begin{bmatrix} \psi_1^2 S^2 & 0 & 0 & 0 & 0 \\ 0 & \psi_2^2 E^2 & 0 & 0 & 0 \\ 0 & 0 & \psi_3^2 I^2 & 0 & 0 \\ 0 & 0 & 0 & \psi_4^2 U^2 & 0 \\ 0 & 0 & 0 & 0 & \psi_5^2 R^2 \end{bmatrix}. \tag{9}$$

Apparently, the matrix $\mathcal{Z}$ is positive definite for any compact subset of $\mathbb{R}_+^5$, so the condition (C.1) in Lemma 1 holds.

To verify the second condition, we establish a $\mathcal{C}^2-$function $D : \mathbb{R}_+^5 \to \mathbb{R}$ as follows:

$$\begin{aligned} D(S,E,I,U,R) =& F(-\ln S - \nu_1 \ln E - \nu_2 \ln I - \nu_3 \ln U - \nu_4 \ln R) - \ln S - \ln E - \ln R \\ &+ \frac{1}{1+\theta}(S+E+I+U+R)^{\theta+1} = FD_1 + D_2 + D_3. \end{aligned} \tag{10}$$

Such that $\nu_1 = \frac{\eta\beta\gamma\sigma^2(1-f)}{\sigma_1^2\sigma_2\sigma_3\sigma_4}$, $\nu_2 = \frac{\eta\beta\gamma\sigma^2(1-f)}{\sigma_1\sigma_2^2\sigma_3\sigma_4}$, $\nu_3 = \frac{\eta\beta\gamma\sigma^2(1-f)}{\sigma_1\sigma_2\sigma_3^2\sigma_4}$ and $\nu_4 = \frac{\eta\beta\gamma\sigma^2(1-f)}{\sigma_1\sigma_2\sigma_3\sigma_4^2}$, and $\theta > 1$ is a contant satisfying $\eta - \frac{\theta}{2}(\psi_1 \vee \psi_2 \vee \psi_3 \vee \psi_4 \vee \psi_5)$. In addition $F > 0$ is a sufficiently large number satisfying $-F\Psi + E_1 \leq -2$, where $\Psi = \frac{\eta\beta\gamma\sigma^2(1-f)}{\hat{\sigma_1}\hat{\sigma_2}\hat{\sigma_3}\hat{\sigma_4}} - (\mu + \frac{\psi_1^2}{2}) > 0$ and

$$E_1 = \sup_{(S,E,I,U,R)\in\mathbb{R}_+^5} \Big\{ -\frac{1}{4}[\mu - \frac{\theta}{2}(\psi_1^2 \vee \psi_2^2 \vee \psi_3^2 \vee \psi_4^2 \vee \psi_5^2)]E^{\theta+1} + 3\mu + \sigma + \frac{\psi_1^2+\psi_2^2+\psi_5^2}{2} + E_2 \Big\}$$

$$\begin{aligned} E_2 = \sup_{(S,E,I,U,R)\in\mathbb{R}_+^5} \Big\{ &\eta(S+E+I+U+R)^\theta - \frac{1}{2}[\mu - \frac{\theta}{2}(\psi_1^2 \vee \psi_2^2 \vee \psi_3^2 \vee \psi_4^2 \vee \psi_5^2)] \\ &\times (S^{\theta+1} + E^{\theta+1} + I^{\theta+1} + U^{\theta+1} + R^{\theta+1}) \Big\}. \end{aligned}$$

$$\begin{aligned} E_3 = \sup_{(S,E,I,U,R)\in\mathbb{R}_+^5} \Big\{ &-\frac{1}{4}[\mu - \frac{\theta}{2}(\psi_1^2 \vee \psi_2^2 \vee \psi_3^2 \vee \psi_4^2 \vee \psi_5^2)](S^{\theta+1} + E^{\theta+1} + I^{\theta+1} + U^{\theta+1} + R^{\theta+1}) \\ &- \frac{1}{4}[\mu - \frac{\theta}{2}(\psi_1^2 \vee \psi_2^2 \vee \psi_3^2 \vee \psi_4^2 \vee \psi_5^2)]E^{\theta+1} - \frac{\eta}{S} + 3\mu + \sigma + \frac{\psi_1^2+\psi_2^2+\psi_5^2}{2} \end{aligned}$$

The function $D(S,E,I,U,R)$ is continuous and $\|(S,E,I,U,R)\| \to \infty$. Thus, it has a minimum point $(S(0),E(0),I(0),U(0),R(0))$ in the interior of $\mathbb{R}_+^5$. Hence, we define $\tilde{D} : \mathbb{R}_+^5 \to \mathbb{R}_+$ as: $\tilde{D} = D(S,E,I,U,R) - D(S(0),E(0),I(0),U(0),R(0))$, where $(S,E,I,U,R) \in (\frac{1}{\rho},\rho) \times (\frac{1}{\rho},\rho) \times (\frac{1}{\rho},\rho) \times (\frac{1}{\rho},\rho) \times (\frac{1}{\rho},\rho)$ and $\rho > 1$ is a sufficiently large integer. Let $D_1 = -\ln S - \nu_1 \ln E -$

$\nu_2 \ln I - \nu_3 \ln U - \nu_4 \ln R$, by applying Itô's formula to D_1, we get

$$\begin{aligned}\mathcal{L}D_1 =& -\frac{\eta}{S} + \beta E + \mu - \nu_1\beta S + \nu_1(\sigma+\mu) - \frac{\nu_2\sigma f E}{I} + \nu_2(\gamma+\mu) - \frac{\nu_3\sigma(1-f)E}{U} \\ &+ \nu_3(\gamma+\mu) - \frac{\nu_4(I+U)}{R} + \nu_4\mu + \frac{\psi_1^2}{2} + \frac{\nu_1\psi_2^2}{2} + \frac{\nu_2\psi_3^2}{2} + \frac{\nu_3\psi_4^2}{2} + \frac{\nu_4\psi_5^2}{2} \\ \leq& -5\sqrt[5]{\eta\beta\gamma\sigma^2(1-f)\nu_1\nu_2\nu_3\nu_4} + \mu + \beta E + \frac{\psi_1^2}{2} + \nu_1(\sigma+\mu+\frac{\psi_2^2}{2}) + \nu_2(\gamma+\mu+\frac{\psi_3^2}{2}) \\ &+ \nu_3(\gamma+\mu+\frac{\psi_4^2}{2}) + \nu_4(\mu+\frac{\psi_5^2}{2}) \leq -\frac{\eta\beta\gamma\sigma^2(1-f)}{\hat{\sigma_1}\hat{\sigma_2}\hat{\sigma_3}\hat{\sigma_4}} + \mu + \beta E + \frac{\psi_1^2}{2} = -\Psi + \beta E.\end{aligned} \tag{11}$$

Let $D_2 = -\ln S - \ln E - \ln R$, one can get

$$\mathcal{L}D_2 = -\frac{\eta}{S} + \beta E + 2\mu - \beta S + (\sigma+\mu) - \frac{\gamma(I+U)}{R} + \frac{\psi_1^2+\psi_2^2+\psi_5^2}{2}. \tag{12}$$

Similarly, assume that $D_3 = \frac{1}{1+\theta}(S+E+I+U+R)^{\theta+1}$, by Itô's formula, we have

$$\begin{aligned}\mathcal{L}D_3 =& (S+E+I+U+R)^{\theta}[\eta - \mu(S+E+I+U+R)] + \frac{\theta}{2}(S+E+I+U+R)^{\theta-1} \\ &\times [\psi_1^2S^2 + \psi_2^2E^2 + \psi_3^2I^2 + \psi_4^2U^2 + \psi_5^2R^2] \\ \leq& (S+E+I+U+R)^{\theta}[\eta - \mu(S+E+I+U+R)] + \frac{\theta}{2}(S+E+I+U+R)^{\theta+1} \\ &\times (\psi_1^2 \vee \psi_2^2 \vee \psi_3^2 \vee \psi_4^2 \vee \psi_5^2) \\ \leq& \eta(S+E+I+U+R)^{\theta} - (S+E+I+U+R)^{\theta+1}(\mu - \frac{\theta}{2}(\psi_1^2 \vee \psi_2^2 \vee \psi_3^2 \vee \psi_4^2 \vee \psi_5^2)) \\ \leq& E_2 - \frac{1}{2}[\mu - \frac{\theta}{2}(\psi_1^2 \vee \psi_2^2 \vee \psi_3^2 \vee \psi_4^2 \vee \psi_5^2)](S+E+I+U+R)^{\theta+1} \\ \leq& E_2 - \frac{1}{2}[\mu - \frac{\theta}{2}(\psi_1^2 \vee \psi_2^2 \vee \psi_3^2 \vee \psi_4^2 \vee \psi_5^2)](S^{\theta+1}+E^{\theta+1}+I^{\theta+1}+U^{\theta+1}+R^{\theta+1}).\end{aligned} \tag{13}$$

From equations (11-13), we obtain

$$\begin{aligned}\mathcal{L}\tilde{D} \leq& -F\Psi + 2F\beta E - \frac{1}{2}[\mu - \frac{\theta}{2}(\psi_1^2 \vee \psi_2^2 \vee \psi_3^2 \vee \psi_4^2 \vee \psi_5^2)](S^{\theta+1}+E^{\theta+1}+I^{\theta+1}+U^{\theta+1}+R^{\theta+1}) \\ &- \frac{\eta}{S} + 3\mu - \beta S + \sigma - \frac{\gamma(I+U)}{R} + \frac{\psi_1^2+\psi_2^2+\psi_5^2}{2} + E_2 \\ \leq& -F\Psi + 2F\beta E - \frac{1}{4}[\mu - \frac{\theta}{2}(\psi_1^2 \vee \psi_2^2 \vee \psi_3^2 \vee \psi_4^2 \vee \psi_5^2)](S^{\theta+1}+E^{\theta+1}+I^{\theta+1}+U^{\theta+1}+R^{\theta+1}) \\ &- \frac{1}{4}[\mu - \frac{\theta}{2}(\psi_1^2 \vee \psi_2^2 \vee \psi_3^2 \vee \psi_4^2 \vee \psi_5^2)]E^{\theta+1} - \frac{\eta}{S} + 3\mu - \beta S + \sigma - \frac{\gamma(I+U)}{R} \\ &+ \frac{\psi_1^2+\psi_2^2+\psi_5^2}{2} + E_2.\end{aligned} \tag{14}$$

For $\epsilon > 0$, we define a bounded set

$$\mathcal{N} = \Big\{(S,E,I,U,R) \in \mathbb{R}_+^5 : \epsilon \leq S \leq \frac{1}{\epsilon}, : \epsilon \leq E \leq \frac{1}{\epsilon}, : \epsilon \leq I \leq \frac{1}{\epsilon}, : \epsilon \leq U \leq \frac{1}{\epsilon}, : \epsilon \leq R \leq \frac{1}{\epsilon^2}\Big\}$$

From Lemma 1, we will verify that $\mathcal{L}\tilde{D} \leq -1$ for all $(S,E,I,U,R) \in \mathbb{R}^5_+ \setminus \mathcal{N}_\epsilon = \cup_{i=1}^{10}\mathcal{N}_i$, where

$$\begin{aligned}
&\mathcal{N}_1 = \{(S,E,I,U,R) \in \mathbb{R}^5_+ : 0 < S < \epsilon\}, \quad \mathcal{N}_2 = \{(S,E,I,U,R) \in \mathbb{R}^5_+ : 0 < E < \epsilon\},\\
&\mathcal{N}_3 = \{(S,E,I,U,R) \in \mathbb{R}^5_+ : 0 < I < \epsilon, R > \epsilon^2\}, \ \mathcal{N}_4 = \{(S,E,I,U,R) \in \mathbb{R}^5_+ : 0 < U < \epsilon, R > \epsilon^2\}\\
&\mathcal{N}_5 = \{(S,E,I,U,R) \in \mathbb{R}^5_+ : 0 < R < \epsilon^2, I > \epsilon\}, \quad \mathcal{N}_6 = \{(S,E,I,U,R) \in \mathbb{R}^5_+ : S > \frac{1}{\epsilon}\}\\
&\mathcal{N}_7 = \{(S,E,I,U,R) \in \mathbb{R}^5_+ : E > \frac{1}{\epsilon}\}, \quad \mathcal{N}_8 = \{(S,E,I,U,R) \in \mathbb{R}^5_+ : I > \frac{1}{\epsilon}\}\\
&\mathcal{N}_9 = \{(S,E,I,U,R) \in \mathbb{R}^5_+ : U > \frac{1}{\epsilon}\}, \quad \mathcal{N}_{10} = \{(S,E,I,U,R) \in \mathbb{R}^5_+ : R > \frac{1}{\epsilon^2}\}
\end{aligned} \tag{15}$$

Let

$$\begin{aligned}
\mathcal{G} =& 2F\beta E - \frac{1}{4}[\mu - \frac{\theta}{2}(\psi_1^2 \vee \psi_2^2 \vee \psi_3^2 \vee \psi_4^2 \vee \psi_5^2)](S^{\theta+1} + E^{\theta+1} + I^{\theta+1} + U^{\theta+1} + R^{\theta+1})\\
&- \frac{1}{4}[\mu - \frac{\theta}{2}(\psi_1^2 \vee \psi_2^2 \vee \psi_3^2 \vee \psi_4^2 \vee \psi_5^2)]E^{\theta+1} + 3\mu + \sigma + \frac{\psi_1^2 + \psi_2^2 + \psi_5^2}{2} + E_2.
\end{aligned} \tag{16}$$

- Case I: for any $(S,E,I,U,R) \in \mathcal{N}_1$, we have

$$\begin{aligned}
\mathcal{L}\tilde{D} \leq& -\frac{\eta}{S} + 2F\beta E - \frac{1}{4}[\mu - \frac{\theta}{2}(\psi_1^2 \vee \psi_2^2 \vee \psi_3^2 \vee \psi_4^2 \vee \psi_5^2)](S^{\theta+1} + E^{\theta+1} + I^{\theta+1} + U^{\theta+1} + R^{\theta+1})\\
&- \frac{1}{4}[\mu - \frac{\theta}{2}(\psi_1^2 \vee \psi_2^2 \vee \psi_3^2 \vee \psi_4^2 \vee \psi_5^2)]E^{\theta+1} + 3\mu - \beta S + \sigma - \frac{\gamma(I+U)}{R}\\
&+ \frac{\psi_1^2 + \psi_2^2 + \psi_5^2}{2} + E_2 \leq -\frac{\eta}{S} + \mathcal{G} \leq -\frac{\eta}{\epsilon} + \mathcal{G} \leq -1,
\end{aligned} \tag{17}$$

  which is obtained as $0 < \epsilon \leq \dfrac{\eta}{\mathcal{G}+1}$.

- Case II: for any $(S,E,I,U,R) \in \mathcal{N}_2$, we have

$$\begin{aligned}
\mathcal{L}\tilde{D} \leq& -F\Psi + 2F\beta E - \frac{1}{4}[\mu - \frac{\theta}{2}(\psi_1^2 \vee \psi_2^2 \vee \psi_3^2 \vee \psi_4^2 \vee \psi_5^2)]E^{\theta+1} + 3\mu + \sigma + \frac{\psi_1^2 + \psi_2^2 + \psi_5^2}{2} + E_2\\
\leq& -F\Psi + 2F\beta E + E_1 \leq -F\Psi + 2F\beta\epsilon + E_1 \leq -1.
\end{aligned} \tag{18}$$

  since $-F\Psi + E_1 \leq -2$ and $-F\Psi + 2F\beta\epsilon + E_1 \leq -1$.

- Case III: for any $(S,E,I,U,R) \in \mathcal{N}_3$, one obtains

$$\begin{aligned}
\mathcal{L}\tilde{D} \leq& -\frac{\gamma I}{R} + 2F\beta E - \frac{1}{4}[\mu - \frac{\theta}{2}(\psi_1^2 \vee \psi_2^2 \vee \psi_3^2 \vee \psi_4^2 \vee \psi_5^2)](S^{\theta+1} + E^{\theta+1} + I^{\theta+1} + U^{\theta+1} + R^{\theta+1})\\
&- \frac{1}{4}[\mu - \frac{\theta}{2}(\psi_1^2 \vee \psi_2^2 \vee \psi_3^2 \vee \psi_4^2 \vee \psi_5^2)]E^{\theta+1} + 3\mu - \beta S + \sigma - \frac{\eta}{S}\\
&+ \frac{\psi_1^2 + \psi_2^2 + \psi_5^2}{2} + E_2 \leq -\frac{\gamma I}{R} + \mathcal{G} \leq -\frac{\gamma}{\epsilon} + \mathcal{G} \leq -1,
\end{aligned} \tag{19}$$

  which is obtained as $0 < \epsilon \leq \dfrac{\gamma}{\mathcal{G}+1}$. In the same manner we can prove Case IV.

- Case V: for any $(S, E, I, U, R) \in \mathcal{N}_5$, we get

$$\begin{aligned}\mathcal{L}\tilde{D} \leq & -\frac{\gamma I}{R} + 2F\beta E - \frac{1}{4}[\mu - \frac{\theta}{2}(\psi_1^2 \vee \psi_2^2 \vee \psi_3^2 \vee \psi_4^2 \vee \psi_5^2)](S^{\theta+1} + E^{\theta+1} + I^{\theta+1} + U^{\theta+1} + R^{\theta+1}) \\ & - \frac{1}{4}[\mu - \frac{\theta}{2}(\psi_1^2 \vee \psi_2^2 \vee \psi_3^2 \vee \psi_4^2 \vee \psi_5^2)]E^{\theta+1} + 3\mu - \beta S + \sigma - \frac{\eta}{S} \\ & + \frac{\psi_1^2 + \psi_2^2 + \psi_5^2}{2} + E_2 \leq -\frac{\gamma}{\epsilon^3} + \mathcal{G} \leq -1,\end{aligned} \tag{20}$$

  which is obtained as $0 < \epsilon \leq \sqrt[3]{\frac{\gamma}{\mathcal{G}+1}}$.

- Case VI: for any $(S, E, I, U, R) \in \mathcal{N}_6$, we have

$$\mathcal{L}\tilde{D} \leq -\frac{\mu}{4}S^{\theta+1} + E_3 \leq E_3 - \frac{\mu}{4}\epsilon^{-(\theta+1)} \leq -1, \tag{21}$$

  where $0 < \epsilon \leq \left[\frac{\mu}{4(E_3+1)}\right]^{1/(\theta+1)}$. Cases VII, VIII and IX are the same as Case VI.

- Case X: for any $(S, E, I, U, R) \in \mathcal{N}_{10}$, one obtains

$$\mathcal{L}\tilde{D} \leq -\frac{\mu}{4}R^{\theta+1} + E_3 \leq E_3 - \frac{\mu}{4}\epsilon^{-2(\theta+1)} \leq -1, \tag{22}$$

  where $0 < \epsilon \leq \left[\frac{\mu}{4(E_3+1)}\right]^{1/2(\theta+1)}$.

Therefore, $\mathcal{L}\tilde{D} \leq -1$ for all $(S, E, I, U, R) \in \mathcal{N}_\epsilon^{10}$. The condition (C.2) of Lemma 1 holds. According to Lemma 1, system (2) has a unique stationary distribution and it has the ergodic property. The conclusion is confirmed. □

Herein, we mainly discuss the extinction of the disease under some conditions. To begin with, we present the following lemmas [17].

Lemma 2. *Assume that (S(0),E(0),I(0),U(0),R(0)) is the initial value for the solution of the model (2), then*

$$\lim_{t\to\infty}\frac{S(t)}{t} = 0, \quad \lim_{t\to\infty}\frac{E(t)}{t} = 0, \quad \lim_{t\to\infty}\frac{I(t)}{t} = 0, \quad \lim_{t\to\infty}\frac{U(t)}{t} = 0, \quad \lim_{t\to\infty}\frac{R(t)}{t} = 0.$$

Lemma 3. *Assume that $\mu > \frac{1}{2}(\psi_1^2 \vee \psi_2^2 \vee \psi_3^2 \vee \psi_4^2 \vee \psi_5^2)$, where (S(t),E(t),I(t),R(t),V(t)) is the solution of (2) with initial value (S(0),E(0),I(0),U(0),R(0)), we have*

$$\begin{aligned}&\lim_{t\to\infty}\frac{\int_0^t S(r)dW_1(r)}{t} = 0, \quad \lim_{t\to\infty}\frac{\int_0^t E(r)dW_2(r)}{t} = 0, \quad \lim_{t\to\infty}\frac{\int_0^t I(r)dW_3(r)}{t} = 0, \\ &\lim_{t\to\infty}\frac{\int_0^t U(r)dW_4(r)}{t} = 0, \quad \lim_{t\to\infty}\frac{\int_0^t R(r)dW_5(r)}{t} = 0.\end{aligned} \tag{23}$$

It is easy to compute that the deterministic system (1) admits a disease-free equilibrium point. For the stochastic system (2), we obtain a sufficient condition on the extinction of disease in the following theorem.

Theorem 3. *Let $(S(t), E(t), I(t), U(t), R(t))$ be a solution of system (2) with initial value (3). If $\mathcal{R}_0^e := \frac{\beta\eta}{\mu(\sigma+\mu+\frac{\psi_2^2}{2})} < 1$ and $\mu > \frac{1}{2}(\psi_1^2 \vee \psi_2^2 \vee \psi_3^2 \vee \psi_4^2 \vee \psi_5^2)$, then the disease will tend to extinction with probability one, and the solution of system (2) satisfies: $\lim_{t\to\infty} S(t) = \frac{\eta}{\mu}$ and $\lim_{t\to\infty} E(t) = \lim_{t\to\infty} I(t) = \lim_{t\to\infty} U(t) = \lim_{t\to\infty} R(t) = 0$.*

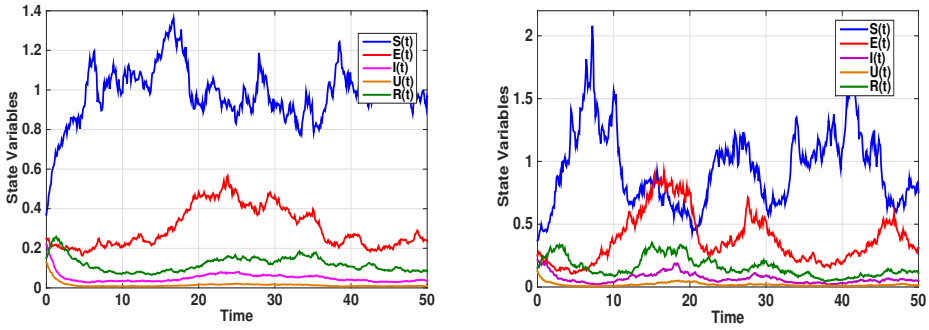

Figure 2: Stationary distribution: the simulation of the path $S(t), E(t), I(t), U(t), R(t)$ for model (2) with $\psi_i = 0.1$, $i = 1, \ldots, 5$ (left); In the right banner, the intensities of white noise increased to $\psi_i = 0.2$, $i = 1, \ldots, 5$ for which the range of the stochastic perturbations increased. However, the disease still persistent as $\mathcal{R}_0^s > 1$.

Proof. By taking the integration of the first equation of system (2), we have

$$\frac{S(t) - S(0)}{t} = \eta - \beta\langle S(t)\rangle\langle E(t)\rangle - \mu\langle S(t)\rangle + \psi_1 \frac{\int_0^t S(r)dW_1(r)}{t}. \tag{24}$$

Therefore,

$$\begin{aligned}\langle S(t)\rangle &= \frac{1}{\mu}[\eta - \beta\langle S(t)\rangle\langle E(t)\rangle - \frac{S(t) - S(0)}{t} + \psi_1 \frac{\int_0^t S(r)dW_1(r)}{t}] \\ &\leq \frac{\eta}{\mu} + \frac{1}{\mu}[\psi_1 \frac{\int_0^t S(r)dW_1(r)}{t} - \frac{S(t) - S(0)}{t}] := \frac{\eta}{\mu} + \Gamma_1(t).\end{aligned} \tag{25}$$

Here, $\lim_{t\to\infty} \Gamma_1(t) = 0$. Using Itô$'s$ formula to the second equation of (2), we obtain:

$$d\log(E(t)) = (\beta S(t) - (\sigma + \mu + \frac{\psi_2^2}{2}))dt + \psi_2 dW_2(t). \tag{26}$$

By taking the integrating of equation (26) from 0 to t, we have

$$\frac{\log E(t)}{t} = \beta\langle S(t)\rangle - (\sigma + \mu + \frac{\psi_2^2}{2}) + \frac{\psi_2 dW_2(t)}{t} + \frac{\log E(0)}{t}, \tag{27}$$

from (25) and (26), we get

$$\begin{aligned}\frac{\log E(t)}{t} &\leq \beta\frac{\eta}{\mu} + \beta\Gamma_1(t) - (\sigma + \mu + \frac{\psi_2^2}{2}) + \frac{\psi_2 dW_2(t)}{t} + \frac{E(0)}{t} \\ &:= \beta\frac{\eta}{\mu} - (\sigma + \mu + \frac{\psi_2^2}{2}) + \beta\Gamma_1(t) + \Gamma_2(t).\end{aligned} \tag{28}$$

Since $\lim_{t\to\infty}\Gamma_2(t)=0$ and if $\mathcal{R}_0^e<1$, we have

$$\lim_{t\to\infty}\frac{\log E(t)}{t}\leq\beta\frac{\eta}{\mu}-(\sigma+\mu):=(\sigma+\mu+\frac{\psi_2^2}{2})(\mathcal{R}_0^e-1)<0. \tag{29}$$

Thus, $\lim_{t\to\infty}E(t)=0$. One may integrate the third, fourth and fifth equations of system (2), we have

$$\begin{aligned}
\frac{I(t)-I(0)}{t}&=\sigma f\langle E(t)\rangle-(\gamma+\mu)\langle I(t)\rangle+\psi_3\frac{\int_0^t I(r)dW_3(r)}{t}.\\
\frac{U(t)-U(0)}{t}&=\sigma(1-f)\langle E(t)\rangle-(\gamma+\mu)\langle U(t)\rangle+\psi_4\frac{\int_0^t U(r)dW_4(r)}{t}.\\
\frac{R(t)-R(0)}{t}&=\gamma\langle I(t)\rangle+\gamma\langle U(t)\rangle-\mu\langle R(t)\rangle+\psi_5\frac{\int_0^t R(r)dW_5(r)}{t}.
\end{aligned} \tag{30}$$

Therefore,

$$\langle I(t)\rangle=\frac{1}{(\gamma+\mu)}[\sigma f\langle E(t)\rangle+\psi_3\frac{\int_0^t I(r)dW_3(r)}{t}-\frac{I(t)-I(0)}{t}]=\frac{\sigma f\langle E(t)\rangle}{(\gamma+\mu)}+\Gamma_3(t). \tag{31}$$

As $\lim_{t\to\infty}\Gamma_3(t)=0$ and if $\mathcal{R}_0^e<1$; then $\lim_{t\to\infty}I(t)=0$. In the same manner we can show that $\lim_{t\to\infty}U(t)=\lim_{t\to\infty}R(t)=0$. Hence, we have

$$\begin{aligned}
d(S(t)+E(t)+I(t)+U(t)+R(t))&=[\eta-\mu(S(t)+E(t)+I(t)+U(t)+R(t))]dt\\
&+\psi_1S(t)dW_1(t)+\psi_2E(t)dW_2(t)+\psi_3I(t)dW_3(t)+\psi_4U(t)dW_4(t)+\psi_5R(t)dW_5(t).
\end{aligned} \tag{32}$$

Thus, we can easily obtain

$$\lim_{t\to\infty}\langle S(t)+E(t)+I(t)+U(t)+R(t)\rangle=\frac{\eta}{\mu}. \tag{33}$$

□

4 Numerical Simulations and Concluding Remarks

We provide some numerical simulations for system (2), utilizing Milstein's higher order method [19], to illustrate the feasibility of our theoretical results. First, we examine the behavior of the stochastic model (2) around $\mathcal{E}^*$ with parameter values $\eta=0.4$; $\beta=0.43905$; $\mu=0.275$; $\sigma=0.15$; $f=0.8$; $\gamma=0.5$ and $\psi_i=0.1$, $i=1,\ldots,5$; such that $\mathcal{R}_0^s>1$, and its corresponding deterministic model. Figure 2 shows the impact of increasing the intensities of white noise, for the which range of the stochastic perturbations increased. However, the disease still persistent as $\mathcal{R}_0^s>1$. Figures 3 and 4 illustrate the extinction criteria in two cases: namely when the stochastic perturbations can eradicate the infectious disease such that $\mathcal{R}_0^e<1$; and the second case when $\mathcal{R}_0<1$ such that the disease is wiped out in the deterministic model, apparently the disease has decreased quickly with the increment of white noise intensity and they all eventually be close to 0. Figure 5 shows the impact of the proportion of exposure to infected f, which signifies that the number of Infected components increase as f increases (left); the the number of unreported individuals decrease as f increases (right).

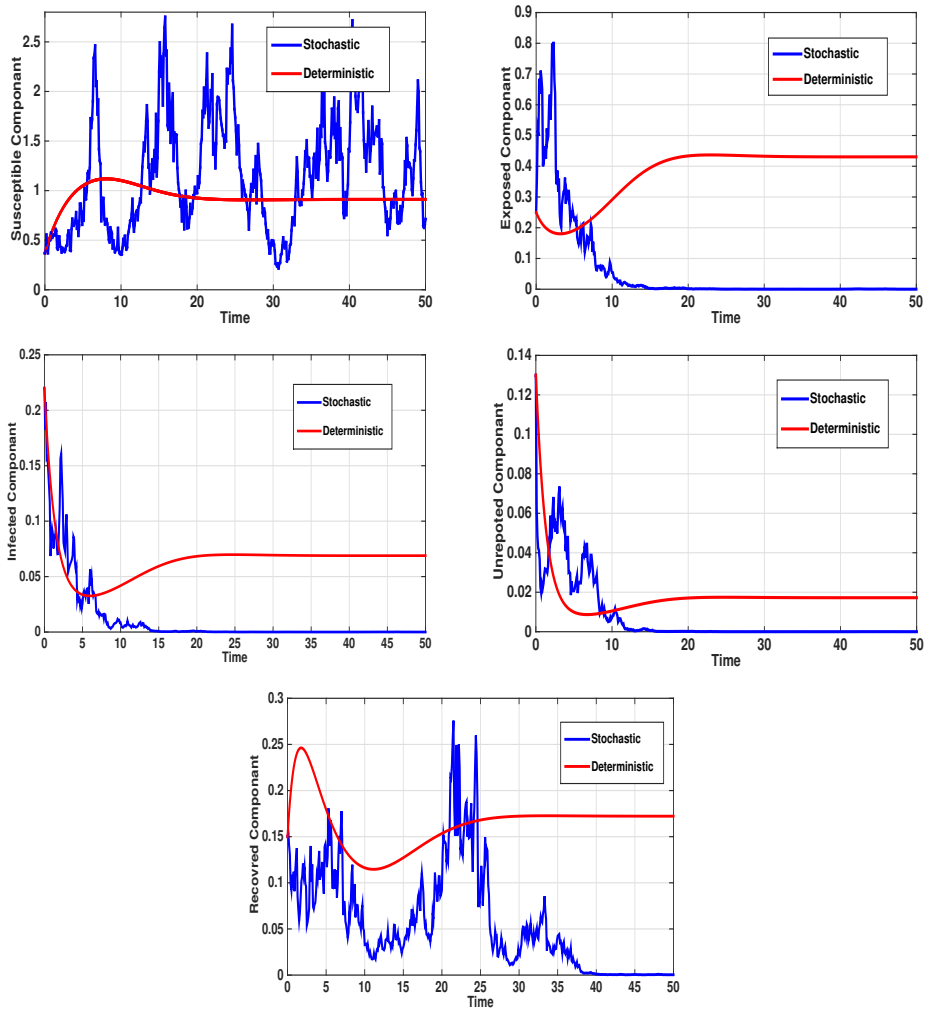

Figure 3: Extinction: the simulation of the path $S(t), E(t), I(t), U(t), R(t)$ for model (2) with $\eta = 0.4$; $\beta = 0.43905$; $\mu = 0.275$; $\sigma = 0.15$; $f = 0.8$; $\gamma = 0.5$ and $\psi_i = 0.3$, $i = 1, \ldots, 5$ and $\psi_i = 0.1$, $i = 1, \ldots, 5$; such that $\mathcal{R}_0^e < 1 < \mathcal{R}_0$, and its corresponding deterministic model. Which illustrates the extinction of the disease when the white noise is relatively large, while the disease is persistent in the undisturbed model.

To check the validity of system (1), we fitted real observations for the number of infected cases of COVID-19 in the UAE from June 28, 2021 to August 10, 2021 using least square approach ([7]). The objective function is defined by

$$\Phi_H(\mathbf{c}) = \sum_{i=1}^{5} \sum_{j=1}^{M} [x^i(t_j, \mathbf{c}) - X_j^i]^2 h_{ij}. \tag{34}$$

The variables S, E, I, U, R are represented by $x^i, i = 1, \ldots, 5$; and model parameters are represented by $\mathbf{c}$. Consequently, we examine the optimum parameter $\hat{\mathbf{c}}$ satisfying $\Phi(\hat{\mathbf{c}}) \leq \min_c \Phi(\mathbf{c}) \equiv$

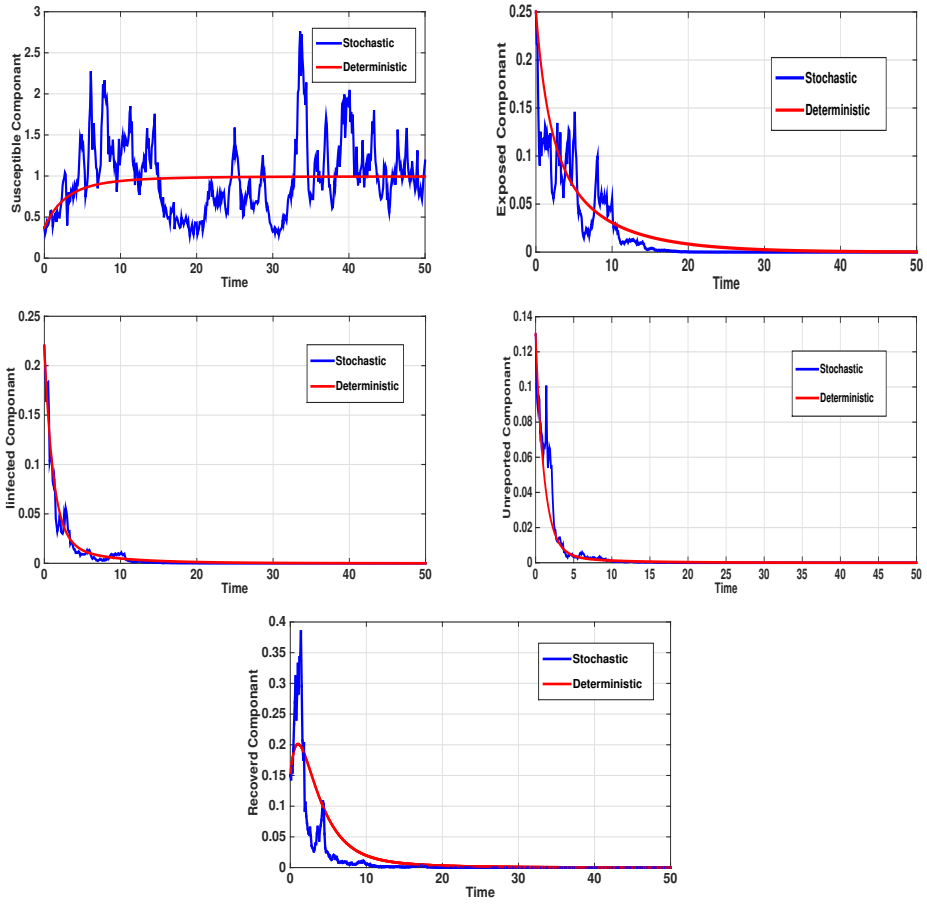

Figure 4: Extinction: the simulation of the path $S(t), E(t), I(t), U(t), R(t)$ for model (2) with $\eta = 0.4$; $\beta = 0.43905$; $\mu = 0.4025$; $\sigma = 0.15$; $f = 0.8$; $\gamma = 0.5$ and $\psi_i = 0.1$, $i = 1, \ldots, 5$; such that $\mathcal{R}_0^e < \mathcal{R}_0 < 1$, and its corresponding deterministic model. the disease vanishes in the stochastic model more rapidly than the undisturbed model because of the effect of stochastic perturbations.

$\max_c \mathcal{L}(\mathbf{c})$. Assume $\mathcal{L}(\mathbf{c})$ is the likelihood function. As a result, estimating the parameters in system (1) is assumed to be an optimization problem; observations is sized in ten thousand increments. The estimated parameters, from the data given in [7], are $\hat{\beta} = 0.99$, $\hat{\mu} = 0.4$, $\hat{\sigma} = 0.6$, so $\tilde{\mathcal{R}}_0 = 1.23 > 1$, see Figure 6.

We arrive at the following remarks.

Remark 1. *According to Theorem 2, the disease will persist in the population (see, Figure 2). From the discussion above, we deduce that the epidemic disappears from the population if the value of the noise is very large confirmed by Theorem 3, for which we deduce the threshold parameter* $\mathcal{R}_0^e := \dfrac{\beta\eta}{\mu(\sigma + \mu + \frac{\psi_2^2}{2})}$ *which less than the reproduction number for the deterministic model. Thus, we can classify the extinction into two cases; case one the infection persists in the deterministic model when* $\mathcal{R}_0 > 1$. *However, the infection dies as in the stochastic model as* $\mathcal{R}_0^e < 1$, *(see Figure 3). Case two; the disease can decrease more rapidly with the increment of*

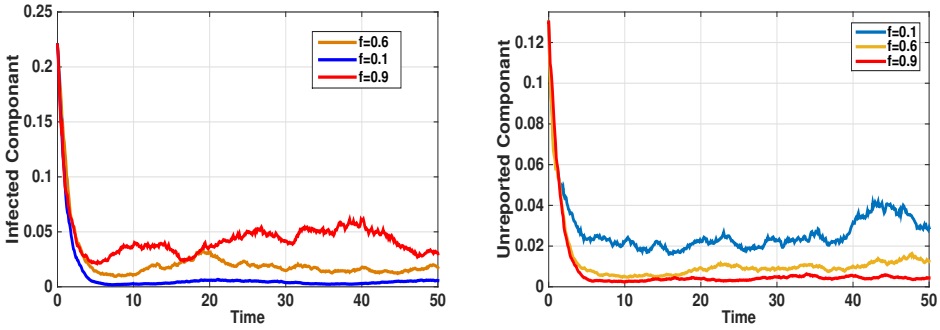

Figure 5: The impact of the proportion of exposure to infected f, which indicates that the number of Infected individuals increase as f increases (left banner); the number of unreported individuals decrease as f increases (right banner).

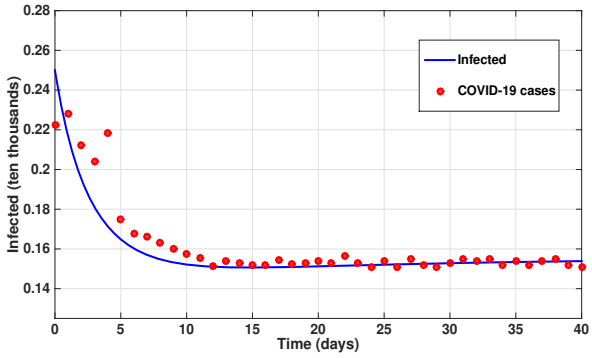

Figure 6: Fitted curve of model (1) and the confirmed COVID-19 cases in the UAE, using least square approach such that $\tilde{\mathcal{R}}_0 = 1.23 > 1$

stochastic disturbance intensity when $\mathcal{R}_0^e < \mathcal{R}_0 < 1$ (see Figure 4). And if the noise is relatively weak, the pandemic persists.

Remark 2. *SEIUR is a reliable model to use when dealing with COVID-19 data. The classical SEIR model is extended here by including cases not officially reported in this study. With the least-squares approach, the model is fitted to real observations in UAE from June 2021 to August 2021. New sufficient conditions for the stability of disease-free and endemic steady states have been derived.*

The effects of environmental noise and more sophisticated models with time delay will be studied in the future [20].

Acknowledgments

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