Computational Coverage of TLG: Nonlinearity*

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Abstract

We study nonlinear connectives (exponentials) in the context of Type Logical Grammar (TLG). We devise four conservative extensions of the displacement calculus with brackets, **Db**!, **Db**!?, **Db**$!_{\mathbf{b}}$ and **Db**$!_{\mathbf{b}}?_{\mathbf{r}}$ which contain the universal and existential exponential modalities of linear logic (**LL**). These modalities do not exhibit the same structural properties as in **LL**, which in TLG are especially adapted for linguistic purposes. The universal modality ! for TLG allows only the commutative and contraction rules, but not weakening, whereas the existential modality ? allows the so-called (intuitionistic) Mingle rule, which derives a restricted version of weakening. We provide a Curry-Howard labelling for both exponential connectives. As it turns out, controlled contraction by ! gives a way to account for the so-called parasitic gaps, and controlled Mingle ? iteration, in particular iterated coordination. Finally, the four calculi are proved to be Cut-Free, and decidability is proved for a linguistically sufficient special case of **Db**$!_{\mathbf{b}}?_{\mathbf{r}}$ (and hence **Db**$!_{\mathbf{b}}$).

1 Introduction

Categorial logic such as displacement calculus **D** [4] is intuitionistic sublinear logic. A major innovation of linear logic are the so-called exponentials which afford a controlled use of structural rules. Here we look at linguistically relevant exponentials in TLG: a universal exponential without weakening in relation to parasitic gaps, and a restriction of the existential exponential to mingle in relation to iterated coordination:

(1) a. man who$_i$ the friends of t_i admire t_i without praising t_i
b. John praises, likes, and will love London.

In Section 2 we define two logically simple calculi **Db**! and **Db**!? with Curry-Howard labelling and we discuss their linguistic suitability. In section 3 we define linguistically refined versions **Db**$!_{\mathbf{b}}$ and **Db**$!_{\mathbf{b}}?_{\mathbf{r}}$, improving the previous calculi in respect of capturing the 'parasicity' of parasitic gaps, that is that, seemingly, parasitic gaps must appear in islands. In Section 4 we discuss Cut-elimination and decidability.

2 Db extended with contraction and mingle modalities

The displacement calculus with brackets **Db** is defined in Figures 1, 2 and 3. The calculus **Db**! is obtained by adding the universal exponential rules in Figure 4. We denote **Db**!? the universal exponential displacement calculus with, in addition, the existential exponential rules of Figure 5.

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$$1.\quad \frac{\Gamma \Rightarrow B\colon \psi \qquad \Delta\langle\overrightarrow{C}\colon z\rangle \Rightarrow D\colon \omega}{\Delta\langle\overrightarrow{C/B}\colon x, \Gamma\rangle \Rightarrow D\colon \omega\{(x\ \psi)/z\}}\,/L \qquad \frac{\Gamma, \overrightarrow{B}\colon y \Rightarrow C\colon \chi}{\Gamma \Rightarrow C/B\colon \lambda y\chi}\,/R$$

$$2.\quad \frac{\Gamma \Rightarrow A\colon \phi \qquad \Delta\langle\overrightarrow{C}\colon z\rangle \Rightarrow D\colon \omega}{\Delta\langle\Gamma, \overrightarrow{A\backslash C}\colon y\rangle \Rightarrow D\colon \omega\{(y\ \phi)/z\}}\,\backslash L \qquad \frac{\overrightarrow{A}\colon x, \Gamma \Rightarrow C\colon \chi}{\Gamma \Rightarrow A\backslash C\colon \lambda x\chi}\,\backslash R$$

$$3.\quad \frac{\Delta\langle\overrightarrow{A}\colon x, \overrightarrow{B}\colon y\rangle \Rightarrow D\colon \omega}{\Delta\langle\overrightarrow{A\bullet B}\colon z\rangle \Rightarrow D\colon \omega\{\pi_1 z/x, \pi_2 z/y\}}\,\bullet L \qquad \frac{\Gamma_1 \Rightarrow A\colon \phi \qquad \Gamma_2 \Rightarrow B\colon \psi}{\Gamma_1, \Gamma_2 \Rightarrow A\bullet B\colon (\phi, \psi)}\,\bullet R$$

$$4.\quad \frac{\Delta\langle\Lambda\rangle \Rightarrow A\colon \phi}{\Delta\langle\overrightarrow{I}\colon x\rangle \Rightarrow A\colon \phi}\,IL \qquad \frac{}{\Lambda \Rightarrow I\colon 0}\,IR$$

Figure 1: Semantically labelled continuous multiplicative rules

$$5.\quad \frac{\Gamma \Rightarrow B\colon \psi \qquad \Delta\langle\overrightarrow{C}\colon z\rangle \Rightarrow D\colon \omega}{\Delta\langle\overrightarrow{C\uparrow_k B}\colon x|_k \Gamma\rangle \Rightarrow D\colon \omega\{(x\ \psi)/z\}}\,\uparrow_k L \qquad \frac{\Gamma|_k \overrightarrow{B}\colon y \Rightarrow C\colon \chi}{\Gamma \Rightarrow C\uparrow_k B\colon \lambda y\chi}\,\uparrow_k R$$

$$6.\quad \frac{\Gamma \Rightarrow A\colon \phi \qquad \Delta\langle\overrightarrow{C}\colon z\rangle \Rightarrow D\colon \omega}{\Delta\langle\Gamma|_k \overrightarrow{A\downarrow_k C}\colon y\rangle \Rightarrow D\colon \omega\{(y\ \phi)/z\}}\,\downarrow_k L \qquad \frac{\overrightarrow{A}\colon x|_k \Gamma \Rightarrow C\colon \chi}{\Gamma \Rightarrow A\downarrow_k C\colon \lambda x\chi}\,\downarrow_k R$$

$$7.\quad \frac{\Delta\langle\overrightarrow{A}\colon x|_k \overrightarrow{B}\colon y\rangle \Rightarrow D\colon \omega}{\Delta\langle\overrightarrow{A\odot_k B}\colon z\rangle \Rightarrow D\colon \omega\{\pi_1 z/x, \pi_2 z/y\}}\,\odot_k L \qquad \frac{\Gamma_1 \Rightarrow A\colon \phi \qquad \Gamma_2 \Rightarrow B\colon \psi}{\Gamma_1|_k \Gamma_2 \Rightarrow A\odot_k B\colon (\Phi, \Psi)}\,\odot_k R$$

$$8.\quad \frac{\Delta\langle 1\rangle \Rightarrow A\colon \phi}{\Delta\langle\overrightarrow{J}\colon x\rangle \Rightarrow A\colon \phi}\,JL \qquad \frac{}{1 \Rightarrow J\colon 0}\,JR$$

Figure 2: Semantically labelled discontinuous multiplicative rules

The very elementary characterisation of (object) relativisation is obtained by assigning a relative pronoun type $(CN\backslash CN)/(S/N)$. This captures the long distance character of relativisation but only allows peripheral extraction. Using the universal exponential we can improve the type assignment to $(CN\backslash CN)/\ (S/!N)$ which, in view of the permutability of the exponential subtype also allows medial extraction.

Various 'islands' can inhibit or block relativisation: weak islands such as subjects (Chomsky 1973[1]) and adverbial phrases, from which extraction is mildly unacceptable, and strong islands such as coordinate structures (Ross 1967[5]) and relative clauses themselves, from which extraction is entirely unacceptable:

(2) a. ?man who$_i$ the friend of t_i laughed
b. ?paper which$_i$ John laughed before reading t_i

(3) a. *man who$_i$ John laughed and Mary likes t_i
b. *man who$_i$ John likes the woman that loves t_i

Furthermore, relativisation can also comprise 'parasitic extraction' in which a relative pronoun binds

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$$15.\quad \frac{\Delta\langle\overrightarrow{A}\colon x\rangle \Rightarrow B\colon\psi}{\Delta\langle[\overrightarrow{[\,]^{-1}A}\colon x]\rangle \Rightarrow B\colon\psi}\,[\,]^{-1}L \qquad \frac{[\Gamma] \Rightarrow A\colon\phi}{\Gamma \Rightarrow [\,]^{-1}A\colon\phi}\,[\,]^{-1}R$$

$$16.\quad \frac{\Delta\langle[\overrightarrow{A}\colon x]\rangle \Rightarrow B\colon\psi}{\Delta\langle\overrightarrow{\langle\rangle A}\colon x\rangle \Rightarrow B\colon\psi}\,\langle\rangle L \qquad \frac{\Gamma \Rightarrow A\colon\phi}{[\Gamma] \Rightarrow \langle\rangle A\colon\phi}\,\langle\rangle R$$

Figure 3: Semantically labelled bracket modality rules

$$17.\quad \frac{\Gamma\langle A\colon x\rangle \Rightarrow B\colon\psi}{\Gamma\langle !A\colon x\rangle \Rightarrow B\colon\psi}\,!L \qquad \frac{!A_1\colon x_1,\ldots,!A_n\colon x_n \Rightarrow A\colon\phi}{!A_1\colon x_1,\ldots,!A_n\colon x_n \Rightarrow !A\colon\phi}\,!R$$

$$\frac{\Delta\langle !A\colon x,\Gamma\rangle \Rightarrow B\colon\psi}{\Delta\langle\Gamma, !A\colon x\rangle \Rightarrow B\colon\psi}\,!P \qquad \frac{\Delta\langle\Gamma, !A\colon x\rangle \Rightarrow B\colon\psi}{\Delta\langle !A\colon x,\Gamma\rangle \Rightarrow B\colon\psi}\,!P$$

$$\frac{\Delta\langle !A_0\colon x_0,\ldots,!A_n\colon x_n, !A_0\colon y_0,\ldots,!A_n\colon y_0\rangle \Rightarrow B\colon\psi}{\Delta\langle !A_0\colon x_0,\ldots,!A_n\colon x_n\rangle \Rightarrow B\colon\psi\{x_0/y_0,\ldots,x_n/y_n\}}\,!C$$

Figure 4: Semantically labelled universal exponential rules

more than one extraction site (Taraldsen 1979[7]; Engdahl 1983[3]; Sag 1983[6]). There must be a 'host' gap which is not in an island, and according to the received wisdom, and according with the terminology 'parasitic', this may license a 'parasitic' gap in (any number of immediate weak) islands:

(4) a. the man who$_i$ the friends of t_i admire t_i
b. the paper which$_i$ John filed t_i without reading t_i
c. the paper which$_i$ the editor of t_i filed t_i without reading t_i

In addition, we observe that these parasitic gaps may in turn function as host gaps licensing further parasitic gaps in (weak) subislands, and so on recursively:

(5) a. man who$_i$ the fact that the friends of t_i admire t_i surprises t_i
b. man who$_i$ the fact that the friends of t_i admire t_i without praising t_i offends t_i without surprising t_i

The bracket modalities of Figure 3 have application to syntactical domains such as prosodic phrases and extraction islands. For example, *walks*: $\langle\rangle N\backslash S$ for the subject condition, and *before*: $[\,]^{-1}(VP\backslash VP)/VP$ for the adverbial island constraint. The relative pronoun type $(CN\backslash CN)/(S/!N)$ respects these island constraints because the brackets induced block association and permutation of the exponential hypothetical subtype into the bracketed domains.

The presence of the contraction rule potentially allows for parasitic extraction, but in fact the islands in which the parasitic gaps are supposed to occur are closed off for the reasons just given. Furthermore the calculus as it stands overgenerates pseudo-parasitic multiple extraction in which 'parasitic' gaps do not occur in islands:

(6) a. * the slave who$_i$ John sold t_i to t_i
b. * the slave who$_i$ John sold t_i t_i

Thus the logic of contraction as it stands precisely both undergenerates and overgenerates parasitic extraction. We fix this in the next section.

Using the existential exponential, ?, we can assign a coordinator type *and*: $(?N\backslash N)/N$ allowing iterated coordination as in *John, Bill, Mary and Suzy*: N, or *and*: $(?(S/N)\backslash(S/N))/(S/N)$ for

18. $$\frac{!\Gamma(A\colon x) \Rightarrow ?B\colon \psi(x)}{!\Gamma(?A\colon z) \Rightarrow ?B\colon \bigoplus_{x \in z} \psi(x)} ?L \qquad \frac{\Gamma \Rightarrow A\colon \phi}{\Gamma \Rightarrow ?A\colon [\phi]} ?R$$

$$\frac{\Gamma \Rightarrow ?A\colon \phi \qquad \Delta \Rightarrow ?A\colon \psi}{\Gamma, \Delta \Rightarrow ?A\colon \phi \oplus \psi} ?M$$

Figure 5: Semantically labelled existential exponential rules

John likes, Mary dislikes, and Bill hates, London (iterated right node raising), and so on.

3 Db extended with restricted modalised contraction and mingle

The calculus $\mathbf{Db!_b}$ is obtained by adding to **Db** the restricted universal exponential rules in Figure 6. Note how now the application of contraction induces a bracketed domain. We denote $\mathbf{Db!_b?_r}$ the restricted universal exponential displacement calculus with, in addition, the existential exponential restricted to only succedent occurrences, and with only the rules of Figure 7.

17. $$\frac{\Gamma\langle A\colon x\rangle \Rightarrow B\colon \psi}{\Gamma\langle !A\colon x\rangle \Rightarrow B\colon \psi} !L \qquad \frac{!A_1\colon x_1, \dots, !A_n\colon x_n \Rightarrow A\colon \phi}{!A_1\colon x_1, \dots, !A_n\colon x_n \Rightarrow !A\colon \phi} !R$$

$$\frac{\Delta\langle !A\colon x, \Gamma\rangle \Rightarrow B\colon \psi}{\Delta\langle \Gamma, !A\colon x\rangle \Rightarrow B\colon \psi} !P \qquad \frac{\Delta\langle \Gamma, !A\colon x\rangle \Rightarrow B\colon \psi}{\Delta\langle !A\colon x, \Gamma\rangle \Rightarrow B\colon \psi} !P$$

$$\frac{\Delta\langle !A_0\colon x_0, \dots, !A_n\colon x_n, [!A_0\colon y_0, \dots, !A_n\colon y_0, \Gamma]\rangle \Rightarrow B\colon \psi}{\Delta\langle !A_0\colon x_0, \dots, !A_n\colon x_n, \Gamma\rangle \Rightarrow B\colon \psi\{x_0/y_0, \dots, x_n/y_n\}} !C_b$$

Figure 6: Semantically labelled restricted universal exponential rules

18. $$\frac{\Gamma \Rightarrow A\colon \phi}{\Gamma \Rightarrow ?A\colon [\phi]} ?R$$

$$\frac{\Gamma \Rightarrow A\colon \phi \qquad \Delta \Rightarrow ?A\colon \psi}{\Gamma, \Delta \Rightarrow ?A\colon [\phi|\psi]} ?M_r$$

Figure 7: Semantically labelled restricted existential exponential rules

In the following subsections we report analyses computer-generated by a categorial parser/theorem-prover CatLog2.

3.1 Parasitic relativisation

As we have remarked subjects are weak islands; accordingly in our CatLog fragment there is no derivation of simple relativization from a subject such as:

(7) **man**+[[**that**+[**the**+**friends**+**of**]+**walk**]] : $CNs(m)$

(Note the strong island double brackets of the relative clause ensuring that it is an island from which parasitic extraction is not possible.) However, a weak island 'parasitic' gap can be licensed by a host gap [7]:

(8) **man**+[[**that**+**the**+**friends**+**of**+**admire**]] : $CNs(m)$

Lexical lookup yields:

(9) $\square CNs(m) : man, [[\blacksquare\forall n([]^{-1}[]^{-1}(CNn\backslash CNn)/\blacksquare((\langle\rangle Nt(n)\sqcap !\blacksquare Nt(n))\backslash Sf)) : \lambda A\lambda B\lambda C[(B\ C)\wedge(A\ C)],$
$\blacksquare\forall n(Nt(n)/CNn) : \iota, \square(CNp/PPof) : friends, \square((\forall n(CNn\backslash CNn)/\blacksquare\exists bNb)\&(PPof/\exists aNa)) :$
$\hat{}(\check{}of, \lambda DD),$
$\square(((\langle\rangle(\exists aNa - \exists gNt(s(g)))\backslash Sf)/\exists aNa) : \hat{}\lambda E\lambda F(Pres\ ((\check{}admire\ E)\ F))]] \Rightarrow CNs(m)$

There is the following derivation, where the use of contraction, involving brackets and, in focused proofs, stoups, corresponds to generating the parasitic gap:

$$Nt(s(m)) \Rightarrow Nt(s(m))$$
$$\blacksquare L: \quad \blacksquare Nt(s(m)) \Rightarrow Nt(s(m))$$
$$\exists R: \quad \blacksquare Nt(s(m)) \Rightarrow \exists aNa \qquad PPof \Rightarrow PPof$$
$$/L: \quad PPof/\exists aNa, \blacksquare Nt(s(m)) \Rightarrow PPof$$
$$\&L: \quad (\forall n(CNn\backslash CNn)/\blacksquare\exists bNb)\&(PPof/\exists aNa), \blacksquare Nt(s(m)) \Rightarrow PPof$$
$$\Box L: \quad \Box((\forall n(CNn\backslash CNn)/\blacksquare\exists bNb)\&(PPof/\exists aNa)), \blacksquare Nt(s(m)) \Rightarrow PPof \qquad CNp \Rightarrow CNp$$
$$/L: \quad CNp/PPof, \Box((\forall n(CNn\backslash CNn)/\blacksquare\exists bNb)\&(PPof/\exists aNa)), \blacksquare Nt(s(m)) \Rightarrow CNp$$
$$\Box L: \quad \Box(CNp/PPof), \Box((\forall n(CNn\backslash CNn)/\blacksquare\exists bNb)\&(PPof/\exists aNa)), \blacksquare Nt(s(m)) \Rightarrow CNp \qquad Nt(p) \Rightarrow Nt(p)$$
$$/L: \quad Nt(p)/CNp, \Box(CNp/PPof), \Box((\forall n(CNn\backslash CNn)/\blacksquare\exists bNb)\&(PPof/\exists aNa)), \blacksquare Nt(s(m)) \Rightarrow Nt(p)$$
$$\forall L: \quad \forall n(Nt(n)/CNn), \Box(CNp/PPof), \Box((\forall n(CNn\backslash CNn)/\blacksquare\exists bNb)\&(PPof/\exists aNa)), \blacksquare Nt(s(m)) \Rightarrow Nt(p)$$
$$\blacksquare L: \quad \blacksquare\forall n(Nt(n)/CNn), \Box(CNp/PPof), \Box((\forall n(CNn\backslash CNn)/\blacksquare\exists bNb)\&(PPof/\exists aNa)), \blacksquare Nt(s(m)) \Rightarrow Nt(p)$$
$$\exists R: \quad \blacksquare\forall n(Nt(n)/CNn), \Box(CNp/PPof), \Box((\forall n(CNn\backslash CNn)/\blacksquare\exists bNb)\&(PPof/\exists aNa)), \blacksquare Nt(s(m)) \Rightarrow \exists aNa$$
$$-R: \quad \blacksquare\forall n(Nt(n)/CNn), \Box(CNp/PPof), \Box((\forall n(CNn\backslash CNn)/\blacksquare\exists bNb)\&(PPof/\exists aNa)), \blacksquare Nt(s(m)) \Rightarrow \exists aNa - \exists gNt(s(g))$$
$$\langle\rangle R: \quad [\blacksquare\forall n(Nt(n)/CNn), \Box(CNp/PPof), \Box((\forall n(CNn\backslash CNn)/\blacksquare\exists bNb)\&(PPof/\exists aNa)), \blacksquare Nt(s(m))] \Rightarrow \langle\rangle(\exists aNa - \exists gNt(s(g))) \qquad Sf \Rightarrow Sf$$
$$\backslash L: \quad [\blacksquare\forall n(Nt(n)/CNn), \Box(CNp/PPof), \Box((\forall n(CNn\backslash CNn)/\blacksquare\exists bNb)\&(PPof/\exists aNa)), \blacksquare Nt(s(m))], \langle\rangle(\exists aNa - \exists gNt(s(g)))\backslash Sf \Rightarrow Sf$$

$$Nt(s(m)) \Rightarrow Nt(s(m))$$
$$\blacksquare L: \quad \blacksquare Nt(s(m)) \Rightarrow Nt(s(m))$$
$$\exists R: \quad \blacksquare Nt(s(m)) \Rightarrow \exists aNa$$

$$/L: \quad [\blacksquare\forall n(Nt(n)/CNn), \Box(CNp/PPof), \Box((\forall n(CNn\backslash CNn)/\blacksquare\exists bNb)\&(PPof/\exists aNa)), \blacksquare Nt(s(m))], (\langle\rangle(\exists aNa - \exists gNt(s(g)))\backslash Sf)/\exists aNa, \blacksquare Nt(s(m)) \Rightarrow Sf$$
$$\Box L: \quad [\blacksquare\forall n(Nt(n)/CNn), \Box(CNp/PPof), \Box((\forall n(CNn\backslash CNn)/\blacksquare\exists bNb)\&(PPof/\exists aNa)), \blacksquare Nt(s(m))], \Box((\langle\rangle(\exists aNa - \exists gNt(s(g)))\backslash Sf)/\exists aNa), \blacksquare Nt(s(m)) \Rightarrow Sf$$
$$!P: \quad [\blacksquare Nt(s(m)); \blacksquare\forall n(Nt(n)/CNn), \Box(CNp/PPof), \Box((\forall n(CNn\backslash CNn)/\blacksquare\exists bNb)\&(PPof/\exists aNa))], \Box((\langle\rangle(\exists aNa - \exists gNt(s(g)))\backslash Sf)/\exists aNa), \blacksquare Nt(s(m)) \Rightarrow Sf$$
$$!P: \quad \blacksquare Nt(s(m)); [\blacksquare Nt(s(m)); \blacksquare\forall n(Nt(n)/CNn), \Box(CNp/PPof), \Box((\forall n(CNn\backslash CNn)/\blacksquare\exists bNb)\&(PPof/\exists aNa))], \Box((\langle\rangle(\exists aNa - \exists gNt(s(g)))\backslash Sf)/\exists aNa) \Rightarrow Sf$$
$$!C: \quad \blacksquare Nt(s(m)); \blacksquare\forall n(Nt(n)/CNn), \Box(CNp/PPof), \Box((\forall n(CNn\backslash CNn)/\blacksquare\exists bNb)\&(PPof/\exists aNa)), \Box((\langle\rangle(\exists aNa - \exists gNt(s(g)))\backslash Sf)/\exists aNa) \Rightarrow Sf$$
$$!L: \quad !\blacksquare Nt(s(m)), \blacksquare\forall n(Nt(n)/CNn), \Box(CNp/PPof), \Box((\forall n(CNn\backslash CNn)/\blacksquare\exists bNb)\&(PPof/\exists aNa)), \Box((\langle\rangle(\exists aNa - \exists gNt(s(g)))\backslash Sf)/\exists aNa) \Rightarrow Sf$$
$$\sqcap L: \quad \langle\rangle Nt(s(m))\sqcap !\blacksquare Nt(s(m)), \blacksquare\forall n(Nt(n)/CNn), \Box(CNp/PPof), \Box((\forall n(CNn\backslash CNn)/\blacksquare\exists bNb)\&(PPof/\exists aNa)), \Box((\langle\rangle(\exists aNa - \exists gNt(s(g)))\backslash Sf)/\exists aNa) \Rightarrow Sf$$
$$\backslash R: \quad \blacksquare\forall n(Nt(n)/CNn), \Box(CNp/PPof), \Box((\forall n(CNn\backslash CNn)/\blacksquare\exists bNb)\&(PPof/\exists aNa)), \Box((\langle\rangle(\exists aNa - \exists gNt(s(g)))\backslash Sf)/\exists aNa) \Rightarrow (\langle\rangle Nt(s(m))\sqcap !\blacksquare Nt(s(m)))\backslash Sf$$
$$\blacksquare R: \quad \blacksquare\forall n(Nt(n)/CNn), \Box(CNp/PPof), \Box((\forall n(CNn\backslash CNn)/\blacksquare\exists bNb)\&(PPof/\exists aNa)), \Box((\langle\rangle(\exists aNa - \exists gNt(s(g)))\backslash Sf)/\exists aNa) \Rightarrow \blacksquare((\langle\rangle Nt(s(m))\sqcap !\blacksquare Nt(s(m)))\backslash Sf)$$

$$CNs(m) \Rightarrow CNs(m)$$
$$\Box L: \quad \Box CNs(m) \Rightarrow CNs(m) \qquad CNs(m) \Rightarrow CNs(m)$$
$$\backslash L: \quad \Box CNs(m), CNs(m)\backslash CNs(m) \Rightarrow CNs(m)$$
$$[\,]^{-1}L: \quad \Box CNs(m), [[\,]^{-1}(CNs(m)\backslash CNs(m))] \Rightarrow CNs(m)$$
$$[\,]^{-1}L: \quad \Box CNs(m), [[[\,]^{-1}[\,]^{-1}(CNs(m)\backslash CNs(m))]] \Rightarrow CNs(m)$$

$$/L: \quad \Box CNs(m), [[[\,]^{-1}[\,]^{-1}(CNs(m)\backslash CNs(m))/\blacksquare((\langle\rangle Nt(s(m))\sqcap !\blacksquare Nt(s(m)))\backslash Sf), \blacksquare\forall n(Nt(n)/CNn), \Box(CNp/PPof), \Box((\forall n(CNn\backslash CNn)/\blacksquare\exists bNb)\&(PPof/\exists aNa)), \Box((\langle\rangle(\exists aNa - \exists gNt(s(g)))\backslash Sf)/\exists aNa)]] \Rightarrow CNs(m)$$
$$\forall L: \quad \Box CNs(m), [[\forall n([\,]^{-1}[\,]^{-1}(CNn\backslash CNn)/\blacksquare((\langle\rangle Nt(n)\sqcap !\blacksquare Nt(n))\backslash Sf)), \blacksquare\forall n(Nt(n)/CNn), \Box(CNp/PPof), \Box((\forall n(CNn\backslash CNn)/\blacksquare\exists bNb)\&(PPof/\exists aNa)), \Box((\langle\rangle(\exists aNa - \exists gNt(s(g)))\backslash Sf)/\exists aNa)]] \Rightarrow CNs(m)$$
$$\blacksquare L: \quad \Box CNs(m), [[\blacksquare\forall n([\,]^{-1}[\,]^{-1}(CNn\backslash CNn)/\blacksquare((\langle\rangle Nt(n)\sqcap !\blacksquare Nt(n))\backslash Sf)), \blacksquare\forall n(Nt(n)/CNn), \Box(CNp/PPof), \Box((\forall n(CNn\backslash CNn)/\blacksquare\exists bNb)\&(PPof/\exists aNa)), \Box((\langle\rangle(\exists aNa - \exists gNt(s(g)))\backslash Sf)/\exists aNa)]] \Rightarrow CNs(m)$$

This delivers the following semantics in which the gap variable is multiply bound:

(10) $\lambda C[(\check{}man\ C) \wedge (Pres\ ((\check{}admire\ C)\ (\iota\ (\check{}friends\ C))))]$

3.2 Iterated coordination

To express the lexical semantics of coordination, including iterated coordination and coordination in various arities, we use two combinators: a non-empty list map apply α^+ and a non-empty list map Φ^n combinator Φ^{n+}. The former is a follows:

(11)
$$\begin{array}{rcl} (\alpha^+\ [x]\ y) & = & [(x\ y)] \\ (\alpha^+\ [x, y|z]\ w) & = & [(x\ w)|(\alpha^+\ [y|z]\ w)] \end{array}$$

The latter is thus:

(12)
$$\begin{array}{rcl} (((\Phi^{n+}\ 0\ and)\ x)\ [y]) & = & [y \wedge x] \\ (((\Phi^{n+}\ 0\ or)\ x)\ [y]) & = & [y \vee x] \\ (((\Phi^{n+}\ 0\ and)\ x)\ [y, z|w]) & = & [y \wedge (((\Phi^{n+}\ 0\ and)\ x)\ [z|w])] \\ (((\Phi^{n+}\ 0\ or)\ x)\ [y, z|w]) & = & [y \vee (((\Phi^{n+}\ 0\ or)\ x)\ [z|w])] \\ ((((\Phi^{n+}\ (s\ n)\ c)\ x)\ y)\ z) & = & (((\Phi^{n+}\ n\ c)\ (x\ z))\ (\alpha^+\ y\ z)) \end{array}$$

Transitive verb phrase iterated coordination:

(13) (crd(28)) [**john**]+[[**praises**+**likes**+**and**+**will**+**love**]]+**london** : $S f$

Lexical insertion yields:

(14) $[\blacksquare Nt(s(m)) : j], [[\square((\langle\rangle\exists gNt(s(g))\backslash S f)/\exists aNa) : \hat{}\lambda A\lambda B(Pres\ ((\check{}praise\ A)\ B)), \square((\langle\rangle\exists gNt(s(g))\backslash S f)/\exists aNa) :$
$\hat{}\lambda C\lambda D(Pres\ ((\check{}like\ C)\ D)), \blacksquare\forall f\forall a((?\blacksquare((\langle\rangle Na\backslash S f)/\exists bNb)\backslash[\,]^{-1}[\,]^{-1}((\langle\rangle Na\backslash S f)/\exists bNb))/\blacksquare((\langle\rangle Na\backslash S f)/\exists bNb)) :$
$(\Phi^{n+}\ (s\ (s\ 0))\ and), \blacksquare\forall a((\langle\rangle Na\backslash S f)/(\langle\rangle Na\backslash S b)) : \lambda E\lambda F(Fut\ (E\ F)), \square((\langle\rangle\exists aNa\backslash S b)/\exists aNa) :$
$\hat{}\lambda G\lambda H((\check{}love\ G)\ H)]],$
$\blacksquare Nt(s(n)) : l \Rightarrow S f$

The coordination combinator semantics is such that:

(15)
$$\begin{array}{l} (((((\Phi^{n+}\ (s\ (s\ 0))\ and)\ x)\ [y, z])\ w)\ u) = \\ ((((\Phi^{n+}\ (s\ 0)\ and)\ (x\ w))\ (\alpha^+\ [y, z]\ w))\ u) = \\ ((((\Phi^{n+}\ (s\ 0)\ and)\ (x\ w))\ [(y\ w), (z\ w)])\ u) = \\ (((\phi^{n+}\ 0\ and)\ ((x\ w)\ u))\ (\alpha^+\ [(y\ w), (z\ w)]\ u)) = \\ (((\phi^{n+}\ 0\ and)\ ((x\ w)\ u))\ [((y\ w)\ u), ((z\ w)\ u)]) = \\ [((y\ w)\ u) \wedge [((z\ w)\ u) \wedge ((x\ w)\ u)]] \end{array}$$

There is the derivation:

$$N8344 \Rightarrow N8344$$
$$N8344 \Rightarrow \exists aNa \quad \exists R$$
$$Nt(s(m)) \Rightarrow Nt(s(m))$$
$$Nt(s(m)) \Rightarrow \exists aNa \quad \exists R$$
$$[Nt(s(m))] \Rightarrow \langle\rangle\exists aNa \quad \langle\rangle R$$
$$Sb \Rightarrow Sb$$
$$[Nt(s(m))], \langle\rangle\exists aNa\backslash Sb \Rightarrow Sb \quad \backslash L$$
$$[Nt(s(m))], (\langle\rangle\exists aNa\backslash Sb)/\exists aNa, N8344 \Rightarrow Sb \quad /L$$
$$[Nt(s(m))], \Box((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), N8344 \Rightarrow Sb \quad \Box L$$
$$\langle\rangle Nt(s(m)), \Box((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), N8344 \Rightarrow Sb \quad \langle\rangle L$$
$$\Box((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), N8344 \Rightarrow \langle\rangle Nt(s(m))\backslash Sb \quad \backslash R$$
$$Nt(s(m)) \Rightarrow Nt(s(m))$$
$$[Nt(s(m))] \Rightarrow \langle\rangle Nt(s(m)) \quad \langle\rangle R$$
$$Sf \Rightarrow Sf$$
$$[Nt(s(m))], \langle\rangle Nt(s(m))\backslash Sf \Rightarrow Sf \quad \backslash L$$
$$[Nt(s(m))], (\langle\rangle Nt(s(m))\backslash Sf)/(\langle\rangle Nt(s(m))\backslash Sb), \Box((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), N8344 \Rightarrow Sf \quad /L$$
$$[Nt(s(m))], \forall a((\langle\rangle Na\backslash Sf)/(\langle\rangle Na\backslash Sb)), \Box((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), N8344 \Rightarrow Sf \quad \forall L$$
$$[Nt(s(m))], \blacksquare\forall a((\langle\rangle Na\backslash Sf)/(\langle\rangle Na\backslash Sb)), \Box((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), N8344 \Rightarrow Sf \quad \blacksquare L$$
$$[Nt(s(m))], \blacksquare\forall a((\langle\rangle Na\backslash Sf)/(\langle\rangle Na\backslash Sb)), \Box((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), \exists bNb \Rightarrow Sf \quad \exists L$$
$$\langle\rangle Nt(s(m)), \blacksquare\forall a((\langle\rangle Na\backslash Sf)/(\langle\rangle Na\backslash Sb)), \Box((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), \exists bNb \Rightarrow Sf \quad \langle\rangle L$$
$$\blacksquare\forall a((\langle\rangle Na\backslash Sf)/(\langle\rangle Na\backslash Sb)), \Box((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), \exists bNb \Rightarrow \langle\rangle Nt(s(m))\backslash Sf \quad \backslash R$$
$$\blacksquare\forall a((\langle\rangle Na\backslash Sf)/(\langle\rangle Na\backslash Sb)), \Box((\langle\rangle\exists aNa\backslash Sb)/\exists aNa) \Rightarrow (\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb \quad /R$$
$$\blacksquare\forall a((\langle\rangle Na\backslash Sf)/(\langle\rangle Na\backslash Sb)), \Box((\langle\rangle\exists aNa\backslash Sb)/\exists aNa) \Rightarrow \blacksquare((\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb) \quad \blacksquare R$$

①

$$N8345 \Rightarrow N8345$$
$$N8345 \Rightarrow \exists aNa \quad \exists R$$
$$Nt(s(m)) \Rightarrow Nt(s(m))$$
$$Nt(s(m)) \Rightarrow \exists gNt(s(g)) \quad \exists R$$
$$[Nt(s(m))] \Rightarrow \langle\rangle\exists gNt(s(g)) \quad \langle\rangle R$$
$$Sf \Rightarrow Sf$$
$$[Nt(s(m))], \langle\rangle\exists gNt(s(g))\backslash Sf \Rightarrow Sf \quad \backslash L$$
$$[Nt(s(m))], (\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa, N8345 \Rightarrow Sf \quad /L$$
$$[Nt(s(m))], \Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), N8345 \Rightarrow Sf \quad \Box L$$
$$[Nt(s(m))], \Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \exists bNb \Rightarrow Sf \quad \exists L$$
$$\langle\rangle Nt(s(m)), \Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \exists bNb \Rightarrow Sf \quad \langle\rangle L$$
$$\Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \exists bNb \Rightarrow \langle\rangle Nt(s(m))\backslash Sf \quad \backslash R$$
$$\Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa) \Rightarrow (\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb \quad /R$$
$$\Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa) \Rightarrow \blacksquare((\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb) \quad \blacksquare R$$
$$N8346 \Rightarrow N8346$$
$$N8346 \Rightarrow \exists aNa \quad \exists R$$
$$Nt(s(m)) \Rightarrow Nt(s(m))$$
$$Nt(s(m)) \Rightarrow \exists gNt(s(g)) \quad \exists R$$
$$[Nt(s(m))] \Rightarrow \langle\rangle\exists gNt(s(g)) \quad \langle\rangle R$$
$$Sf \Rightarrow Sf$$
$$[Nt(s(m))], \langle\rangle\exists gNt(s(g))\backslash Sf \Rightarrow Sf \quad \backslash L$$
$$[Nt(s(m))], (\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa, N8346 \Rightarrow Sf \quad /L$$
$$[Nt(s(m))], \Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), N8346 \Rightarrow Sf \quad \Box L$$
$$[Nt(s(m))], \Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \exists bNb \Rightarrow Sf \quad \exists L$$
$$\langle\rangle Nt(s(m)), \Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \exists bNb \Rightarrow Sf \quad \langle\rangle L$$
$$\Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \exists bNb \Rightarrow \langle\rangle Nt(s(m))\backslash Sf \quad \backslash R$$
$$\Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa) \Rightarrow (\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb \quad /R$$
$$\Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa) \Rightarrow \blacksquare((\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb) \quad \blacksquare R$$
$$\Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa) \Rightarrow ?\blacksquare((\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb) \quad ?R$$
$$\Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa) \Rightarrow ?\blacksquare((\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb) \quad ?E$$

②

$$\dfrac{\dfrac{\dfrac{}{\boxed{Nt(s(n))} \Rightarrow Nt(s(n))}}{\boxed{\blacksquare Nt(s(n))} \Rightarrow Nt(s(n))}\blacksquare L}{\blacksquare Nt(s(n)) \Rightarrow \boxed{\exists bNb}}\exists R$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\boxed{Nt(s(m))} \Rightarrow Nt(s(m))}}{\boxed{\blacksquare Nt(s(m))} \Rightarrow Nt(s(m))}\blacksquare L}{[\blacksquare Nt(s(m))] \Rightarrow \boxed{\langle\rangle Nt(s(m))}}\langle\rangle R \qquad \dfrac{}{\boxed{Sf} \Rightarrow Sf}}{[\blacksquare Nt(s(m))], \boxed{\langle\rangle Nt(s(m))\backslash Sf} \Rightarrow Sf}\backslash L$$

$$\dfrac{\blacksquare Nt(s(n)) \Rightarrow \exists bNb \qquad [\blacksquare Nt(s(m))], \langle\rangle Nt(s(m))\backslash Sf \Rightarrow Sf}{[\blacksquare Nt(s(m))], \boxed{(\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb}, \blacksquare Nt(s(n)) \Rightarrow Sf}/L$$

$$\dfrac{}{[\blacksquare Nt(s(m))], [\boxed{[]^{-1}((\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb)}], \blacksquare Nt(s(n)) \Rightarrow Sf}[]^{-1}L$$

$$\dfrac{}{[\blacksquare Nt(s(m))], [[\boxed{[]^{-1}[]^{-1}((\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb)}]], \blacksquare Nt(s(n)) \Rightarrow Sf}[]^{-1}L$$

$$\dfrac{② \qquad [\blacksquare Nt(s(m))], [[[]^{-1}[]^{-1}((\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb)]], \blacksquare Nt(s(n)) \Rightarrow Sf}{[\blacksquare Nt(s(m))], [[\Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \boxed{?\blacksquare((\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb)\backslash[]^{-1}[]^{-1}((\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb)}]], \blacksquare Nt(s(n)) \Rightarrow Sf}\backslash L$$

$$\dfrac{① \qquad \cdots}{[\blacksquare Nt(s(m))], [[\Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \boxed{(?\blacksquare((\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb)\backslash[]^{-1}[]^{-1}((\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb))/\blacksquare((\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb)}, \blacksquare\forall a((\langle\rangle Na\backslash Sf)/(\langle\rangle Na\backslash Sb)), \Box((\langle\rangle\exists aNa\backslash Sb)/\exists aNa)]], \blacksquare Nt(s(n)) \Rightarrow Sf}/L$$

$$\dfrac{}{[\blacksquare Nt(s(m))], [[\Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \boxed{\forall a((?\blacksquare((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash[]^{-1}[]^{-1}((\langle\rangle Na\backslash Sf)/\exists bNb))/\blacksquare((\langle\rangle Na\backslash Sf)/\exists bNb))}, \blacksquare\forall a((\langle\rangle Na\backslash Sf)/(\langle\rangle Na\backslash Sb)), \Box((\langle\rangle\exists aNa\backslash Sb)/\exists aNa)]], \blacksquare Nt(s(n)) \Rightarrow Sf}\forall L$$

$$\dfrac{}{[\blacksquare Nt(s(m))], [[\Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \boxed{\forall f\forall a((?\blacksquare((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash[]^{-1}[]^{-1}((\langle\rangle Na\backslash Sf)/\exists bNb))/\blacksquare((\langle\rangle Na\backslash Sf)/\exists bNb))}, \blacksquare\forall a((\langle\rangle Na\backslash Sf)/(\langle\rangle Na\backslash Sb)), \Box((\langle\rangle\exists aNa\backslash Sb)/\exists aNa)]], \blacksquare Nt(s(n)) \Rightarrow Sf}\forall L$$

$$\dfrac{}{[\blacksquare Nt(s(m))], [[\Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \boxed{\blacksquare\forall f\forall a((?\blacksquare((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash[]^{-1}[]^{-1}((\langle\rangle Na\backslash Sf)/\exists bNb))/\blacksquare((\langle\rangle Na\backslash Sf)/\exists bNb))}, \blacksquare\forall a((\langle\rangle Na\backslash Sf)/(\langle\rangle Na\backslash Sb)), \Box((\langle\rangle\exists aNa\backslash Sb)/\exists aNa)]], \blacksquare Nt(s(n)) \Rightarrow Sf}\blacksquare L$$

All this assigns the correct semantics:

(16) $[(\mathit{Pres}\ ((\check{}\mathit{praise}\ l)\ j)) \wedge [(\mathit{Pres}\ ((\check{}\mathit{like}\ l)\ j)) \wedge (\mathit{Fut}\ ((\check{}\mathit{love}\ l)\ j))]]$

4 Cut elimination and decidability (proof idea)

Cut elimination has several key steps and commutative steps. Here we consider only the key step concerning the existential exponential modality. As usual, the proof proceeds by a double induction on the size of the Cut formula and the sum of the heights of the premises of the Cut occurrences. The so-called pseudo-key step of a right application of ! or $!_b$ (as left premise of Cut) and a contraction (as right premise of Cut) is more involved but still standard.[1] Notice that crucially, the $!_b$-contraction must be defined for $!_b$-modalized sequences as is the case in Figure 6.[2]

The key Cut steps involve the structural rules $!C$ and $?M$. The case of $!C$ is standard in the literature of linear logic; we therefore omit it. What is really new is the $!M$ key Cut, which is as follows. (This key step simply does not exist in the case of the calculi $\mathbf{Db!_b}$, nor in $\mathbf{Db!_b?_r}$ because there are only succedent occurrences of the existential exponential.) Where $!\Delta(\Gamma_i) = !\Delta_1, \Gamma_i, !\Delta_2$, we have that the following rule $?GM$:

(17) $$\frac{!\Delta(\Gamma_1) \Rightarrow ?A \qquad !\Delta(\Gamma_2) \Rightarrow ?A}{!\Delta(\Gamma_1, \Gamma_2) \Rightarrow ?A}\ ?GM$$

is derivable from $?M$ by application of $?M$ and the permutation and contraction !-steps without the use of Cut. Then there is the key step:

$$\dfrac{\dfrac{!\Delta(\Gamma_1) \Rightarrow ?A \qquad !\Delta(\Gamma_2) \Rightarrow ?A}{!\Delta(\Gamma_1, \Gamma_2) \Rightarrow ?A}\ ?GM \qquad \dfrac{}{!\Theta(?A) \Rightarrow ?B}\ ?L}{!\Theta(!\Delta(\Gamma_1, \Gamma_2)) \Rightarrow ?B}\ \mathit{Cut}$$

$$\rightsquigarrow \dfrac{\dfrac{!\Delta(\Gamma_1) \Rightarrow ?A \qquad !\Theta(?A) \Rightarrow ?B}{!\Theta(!\Delta(\Gamma_1)) \Rightarrow ?B}\ \mathit{Cut} \qquad \dfrac{!\Delta(\Gamma_2) \Rightarrow ?A \qquad !\Theta(?A) \Rightarrow ?B}{!\Theta(!\Delta(\Gamma_2)) \Rightarrow ?B}\ \mathit{Cut}}{!\Theta(!\Delta(\Gamma_1, \Gamma_2)) \Rightarrow ?B}\ ?GM$$

Let us see now the proof that the generalized Mingle rule for ? is Cut-free derivable in **Db**!? using !-contractions and ?-Mingle. If we write $!\Delta(\Sigma)$ as $!\Delta_1, \Sigma, !\Delta_2$ for arbitrary configurations Δ_i and Σ, we have the following ?-Mingle derivation:

$$\frac{!\Delta(\Gamma_1) \Rightarrow ?A \qquad !\Delta(\Gamma_1) \Rightarrow ?A}{\mathcal{S} := !\Delta(\Gamma_1), !\Delta(\Gamma_2) \Rightarrow ?A}\ ?M$$

To the end-sequent $\mathcal{S}$ of the above derivation we apply a finite number of !-permutation steps and we get the provable sequent:

$$!\Delta_1, !\Delta_1, \Gamma_1, \Gamma_2, !\Delta_2, !\Delta_2 \Rightarrow ?A$$

Finally, to the above sequent we apply a finite number of !-contraction steps obtaining:

$$!\Delta_1, \Gamma_1, \Gamma_2, !\Delta_2 \Rightarrow ?A$$

[1] Recall that both ! and $!_b$ allow only contraction. No weakening nor expansion are associated to these connectives.

[2] The so-called full Lambek calculus with contraction enjoys Cut-elimination if the contraction rule is generalized to sequences of types.

This last sequent can be written as:

$$!\Delta(\Gamma_1, \Gamma_2) \Rightarrow ?A$$

Hence, the ?-GM structural rule is Cut-free derivable in **Db**!?. This ends the proof idea of the Cut admissibility of the four calculi we have considered.

In order to prove the decidability of a linguistically sufficient special case of $\mathbf{Db!_b?_r}$, which we call *polar bracket non-negative* $\mathbf{Db!_b?_r}$ we introduce two useful technical tools: *bracket-count* of a type and *degree of contraction* of a sequent $\mathcal{S}$. Building upon ([8]), we define the bracket-count of a sequent recursively as follows:

(18) **Definition** (*Bracket-count*)

Where A and B are arbitrary $\mathbf{Db!_b?_r}$-types:

$$\begin{array}{lcl}
\#_{[]}(A) & = & 0 \text{ if } A \text{ is atomic} \\
\#_{[]}(A\bullet B) & = & \#_{[]}(A) + \#_{[]}(B) \\
\#_{[]}(A\odot_i B) & = & \#_{[]}(A) + \#_{[]}(B) \\
\#_{[]}(B/A) & = & \#_{[]}(B) - \#_{[]}(A) \\
\#_{[]}(B\uparrow_k A) & = & \#_{[]}(B) - \#_{[]}(A) \\
\#_{[]}(A\backslash B) & = & \#_{[]}(B) - \#_{[]}(A) \\
\#_{[]}(B\downarrow_k A) & = & \#_{[]}(B) - \#_{[]}(A) \\
\#_{[]}(\langle\rangle A) & = & \#_{[]}(A) + 1 \\
\#_{[]}([\,]^{-1} A) & = & \#_{[]}(A) - 1 \\
\#_{[]}(!A) & = & \#_{[]}(A) \\
\#_{[]}(?A) & = & \#_{[]}(A)
\end{array}$$

Where Δ, Δ_i ($i = 1, \cdots, n, n > 0$) are $\mathbf{Db!_b?_r}$-configurations:

$$\begin{array}{lcl}
\#_{[]}(\Lambda) & = & 0 \\
\#_{[]}(A, \Delta) & = & \#_{[]}(A) + \#_{[]}(\Delta) \\
\#_{[]}(1) & = & 0 \\
\#_{[]}(A\{\Delta_1 : \cdots : \Delta_n\}, \Delta) & = & \sum_{i=1}^{n} \#_{[]}(\Delta_i) + \#_{[]}(\Delta) \\
\#_{[]}([\Delta]) & = & \#_{[]}(\Delta) + 1
\end{array}$$

(19) **Definition** (*Degree of Contraction*)

We define the *degree of contraction* of a sequent $\mathcal{S}{:=}\Delta \Rightarrow A$, $d_c(\mathcal{S})$, in terms of bracket counts as follows:

$$d_c(\mathcal{S}) \stackrel{def}{=} \#_{[]}(A) - \#_{[]}(\Delta)$$

We see now some simple facts on the degree of contraction of sequents:

- **Fact 1**: Given a derivation whose last rule is a binary or unary bracket rule with conclusion $\mathcal{S}$ and premises $\mathcal{S}_i$:

$$d_c(S) \geq d_c(S_i)$$

- **Fact 2**: Suppose that the last rule of a derivation is the contraction rule where the configuration $!\Gamma$ is a bracket-free configuration:

$$\frac{\mathcal{S}_2 := \Delta\langle !\Gamma, [!\Gamma, \Theta]\rangle \Rightarrow A}{\mathcal{S}_1 := \Delta\langle !\Gamma, \Theta\rangle \Rightarrow A}\ !C_b$$

Then we have:

$$d_c(\mathcal{S}_1) > d_c(\mathcal{S}_2)$$

- **Fact 3**: Suppose that the last rule of a derivation is the restricted Mingle rule, where all type-occurrences are bracket-free:

$$\frac{\mathcal{S}_2 := \Delta_1 \Rightarrow A \qquad \mathcal{S}_3 := \Delta_2 \Rightarrow ?A}{\mathcal{S}_1 := \Delta_1, \Delta_2 \Rightarrow ?A}\ ?M_r$$

Then we have:

$$d_c(\mathcal{S}_1) \geq d_c(\mathcal{S}_2) + d_c(\mathcal{S}_3)$$

Finally, a useful arithmetic tool is the length of an arbitrary sequent $\mathcal{S} := \Delta \Rightarrow A$, $|\mathcal{S}|$. The well known length of a type, which is simply its number of connectives, and the (overloaded) length of a configuration Δ, $|\Delta|$, which is the sum of the lengths of all its type-occurrences, we define $|\mathcal{S}|$ as $|\Delta| + |A|$. We have the following theorem:

(20) The Cut-free proof-search space in $\mathbf{Db!_b?_r}$ is finite.

Proof. Let $\prec_{\text{Lex-}\mathbb{N}^2}$ be the total strict lexicographical order in $\mathbb{N}^2$. Consider a sequent $\mathcal{S}$ such that $d_c(\mathcal{S}) \geq 0$ (for otherwise it could not be provable). We want to check its provability. We can expand the current goal sequent $\mathcal{S}$ of the proof-search space **ProofSearch** by a finite number of goal sequents, which can be either the subgoals of a logical rule or a structural rule. We associate to each sequent $\mathcal{S}$ of **ProofSearch** its *measure* $\mu(\mathcal{S}) \overset{def}{=} (d_c(\mathcal{S}), |\mathcal{S}|)$. If we expand $\mathcal{S}$ with a contraction rule, the degree of contraction is strictly decreased. In case of a restricted Mingle rule or a logical rule the degree of contraction may be decreased or remain equal. In case that the degree of contraction remains equal, the lengths of the premises of the applied rule are strictly decreased. Hence, **ProofSearch** is a finitely branched tree such that any path $(S_i)_{i>0}$ of it satisfies $\mu(\mathcal{S}_{i+1}) \prec_{\text{Lex-}\mathbb{N}^2} \mu(\mathcal{S}_i)$ for all i. Since $\prec_{\text{Lex-}\mathbb{N}^2}$ is well-founded every strictly decreasing sequence is finite. Therefore, by König's lemma, **ProofSearch** is finite. □

From the preceding theorem, it follows that $\mathbf{Db!_b?_r}$ is decidable in the case that the exponential subtypes are bracket-free in the sense of not containing bracket modalities within exponentials which give rise to antecedent antibracket modalities nor succedent bracket modalities . We call the restriction to such types polar bracket non-negative $\mathbf{Db!_b?_r}$.

Whether the calculus **Db**!? is decidable is an open problem. However, it is interesting to notice that **Db**!? extended with additive connectives is undecidable. In fact, the Lambek Calculus with additives and the connective !, of which **Db**!? with additives is a conservative extension, is already undecidable. This can be proved by a Girard-style translation $(\cdot)^\bullet$ between the full Lambek calculus with contraction (**FLC**) and the full Lambek calculus with !-contraction (**FLC**!) as follows:

$$\begin{array}{lcl}
A^\bullet & = & A \text{ if } A \text{ is atomic} \\
(B/A)^\bullet & = & B^\bullet/!A^\bullet \\
(A\backslash B)^\bullet & = & !A^\bullet\backslash B^\bullet \\
(A \oplus B)^\bullet & = & !A^\bullet \oplus !B^\bullet \\
(A\& B)^\bullet & = & A^\bullet \& B^\bullet \\
(\Delta \Rightarrow A)^\bullet & = & !\Delta^\bullet \Rightarrow A^\bullet
\end{array}$$

We can prove the following theorem:

(21) **Theorem** (*Embedding translation between* **FLC** *and* **FLC**!)

FLC $\vdash \Delta \Rightarrow A$ iff **FLC**!$(\Delta \Rightarrow A)^\bullet$

(22) **Corollary** (*Undecidability of* **FLC**!)

It has been proved that **FLC** is undecidable [2]. If **FLC**! were decidable, for any **FLC**-sequent $\Delta \Rightarrow A$, we could decide whether its translation $(\Delta \Rightarrow A)^{\bullet}$ is provable. We would have then that **FLC** is decidable. Contradiction.

Appendix: ?-Mingle vs. ?-Expansion

Consider the following structural rule called ?-expansion. For any type A:

(23) $$\frac{\Delta\langle ?A\rangle \Rightarrow B}{\Delta\langle ?A, ?A\rangle \Rightarrow B}\,E$$

It is straightforward to see that **Db**!? + *Cut* is deductively equivalent to **Db**!? + $Exp - Mingle + Cut$. However, **Db**!? + *Cut* enjoys Cut elimination, but **Db**!? + $Exp - Mingle + Cut$ does not enjoy Cut elimination.

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