On the constant domains principle and its weakened versions in the Kripke sheaf semantics

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We consider superintuitionistic predicate logics understood in the usual way, as sets of predicate formulas (without function symbols) containing all axioms of Heyting predicate logic **Q-H** and closed under modus ponens, generalization, and substitution of arbitrary formulas for atomic ones (we are mainly interested in logics without equality, and only sometimes we mention logics with equality).

1 We consider the semantics of predicate Kripke frames with equality (called I-frames, for short), which is equivalent to the semantics of Kripke sheaves (see [3]). Namely, a *(predicate) Kripke frame* is a pair $M=(W,U)$ formed by a poset W with the least element 0_W and a domain map U defined on W such that $U(u)\subseteq U(v)$ for $u\leq v$. An *I-frame* is a triple $M=(W,U,I)$, in which (W,U) is a Kripke frame and I is a family of equivalence relations I_u on $U(u)$ for $u\in W$ such that $I_u\subseteq I_v$ for $u\leq v$.

A *valuation* $u\vDash A$ (for $u\in W$ and formulas A with parameters replaced by elements of $U(u)$) satisfies the monotonicity: $u\leq v,\ u\vDash A\ \Rightarrow\ v\vDash A$
and the usual inductive clauses for connectives and quantifiers, e.g.
$u\vDash(B\to C)\ \Leftrightarrow\ \forall v\geq u\,[(v\vDash B)\Rightarrow(v\vDash C)]$,
$u\vDash\forall x\,B(x)\ \Leftrightarrow\ \forall v\geq u\,\forall c\in U(v)\,[v\vDash B(c)]$,
etc. (for the case with equality, $a=b$ is interpreted by aI_ub in an I-frame and by $a=b$ in a usual Kripke frame, for $a,b\in U(u)$). For an I-frame we admit only the valuations preserving I_u (on every $U(u), u\in W$), i.e.,

$$\bigwedge_i (a_i\,I_u\,b_i)\ \Rightarrow\ (\,u\vDash A(a_1,\ldots,a_n)\ \Leftrightarrow\ u\vDash A(b_1,\ldots,b_n)\,).$$

A formula $A(\mathbf{x})$ is *valid* in M if it is true under any valuation in M, i.e., if $u\vDash A(\mathbf{a})$ for any $u\in W$ and $\mathbf{a}\in(D_u)^n$. The *predicate logic* $\mathbf{L}(M)$ *of an (I-)frame* M is the set of all formulas valid in M.

We consider the constant domains principle

$$D=\forall x(P(x)\vee Q)\to\forall xP(x)\vee Q$$

(where P and Q are unary and 0-ary symbols, respectively),
and two its weakened versions, namely:

$$D^-=\forall x\,(\neg P(x)\vee Q)\to\forall x\neg P(x)\vee Q\,,\qquad\text{and}$$

$$D^*=\forall x(P(x)\vee Q)\to Q\vee\forall x\exists y\,(P(y)\&\neg\neg[R(x,x)\to R(x,y)])\,;$$

here D^* simulates the following formula with equality:

$$D^*_{=}=\forall x(P(x)\vee Q)\to Q\vee\forall x\exists y(P(y)\&\neg\neg[x=y])$$

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(i.e., $[\mathbf{Q}\text{-}\mathbf{H}^{=}+D^*] = [\mathbf{Q}\text{-}\mathbf{H}^{=}+D^*_{=}]$). The formula D states (in an (I-)frame) that $U(u)=U(0_W)$ for every $u\in W$, i.e., $\forall a\in U(u)\,[\,a\in U(0_W)\,]$. Now, D^* states that

any individual $a\in U(u)$ has a $\neg\neg$-ancestor in 0_W,

i.e., there exists $b\in U(0_W)$ such that $u\vDash\neg\neg(a=b)$.

Similarly, D^- states that $\forall a\in U(u)\,\exists b\in U(0_W)\,[\,u\nvDash\neg(a=b)\,]$.

Clearly, $D\vdash D^*\vdash D^-$ (we write $A\vdash B$ for $[\mathbf{Q}\text{-}\mathbf{H}+A]\vdash B$).

The logic $[\mathbf{Q}\text{-}\mathbf{H}+D^-]$ is incomplete w.r.t. I-frames (as well as some its natural extensions); its completion (which is not finitely axiomatizable) is described in [5]. On the other hand, [7] proved the completeness w.r.t. I-frames (i.e., the Kripke sheaf completeness) for the following logics: $[\mathbf{Q}\text{-}\mathbf{H}+D^*]$, $[\mathbf{Q}\text{-}\mathbf{H}+D^*\&K]$, $[\mathbf{Q}\text{-}\mathbf{H}+D^*\&K\&J]$,
and the incompleteness for $[\mathbf{Q}\text{-}\mathbf{H}+D^*\&J]$,

where $K=\neg\neg\forall x\,(P(x)\vee\neg P(x))$ (Kuroda's formula)

and $J=\neg Q\vee\neg\neg Q$ (weak excluded middle).

This shows that in some sense the axiom D^* for the Kripke sheaf semantics behaves like the axiom D for the usual Kripke semantics; recall that the logics $[\mathbf{Q}\text{-}\mathbf{H}+D]$, $[\mathbf{Q}\text{-}\mathbf{H}+D\&K]$, and $[\mathbf{Q}\text{-}\mathbf{H}+D\&K\&J]$ are Kripke-complete, while $[\mathbf{Q}\text{-}\mathbf{H}+D\&J]$ is Kripke-incomplete.

And here we claim that this similarity is not too close.

Recall that $[\mathbf{H}+P_2\&J]$ is the greatest non-classical superintuitionistic propositional logic (here $P_2=[\,Q_0\vee(Q_0\to Q\vee\neg Q)\,]$ is the axiom of height 2).

Lemma (Main Lemma). $[\mathbf{Q}\text{-}\mathbf{H}+P_2\&J+D^*]\nvdash K$.

On the other hand, clearly, K is valid in every (I-)frame of a finite height. Therefore,

Theorem. Let $[\mathbf{Q}\text{-}\mathbf{H}+\mathbf{\Lambda}]\subseteq\mathbf{L}\subseteq[\mathbf{Q}\text{-}\mathbf{H}+P_2\&J+D^*]$ for a superintuitionistic propositional logic $\mathbf{\Lambda}$ of a finite slice. Then $\mathbf{L}$ is Kripke sheaf incomplete.

Corollary. The logics $[\mathbf{Q}\text{-}\mathbf{H}+\mathbf{\Lambda}+D^*]$ and $[\mathbf{Q}\text{-}\mathbf{H}+\mathbf{\Lambda}+D^-]$ are Kripke sheaf incomplete for every non-classical superintuitionistic propositional logic $\mathbf{\Lambda}$ of a finite slice.

Recall that the logics $[\mathbf{Q}\text{-}\mathbf{H}+\mathbf{\Lambda}+D]$ are Kripke-complete e.g. for all $\mathbf{\Lambda}$ being tabular logics or subframe logics (i.e., logics axiomatizable by $\to$-formulas) (see [4, Theorems 3.7 and 3.9]), while the logics $[\mathbf{Q}\text{-}\mathbf{H}+\mathbf{\Lambda}]$ are Kripke-incomplete for all non-classical $\mathbf{\Lambda}$ of finite slices [2, Theorem 3.2] (and moreover, they are Kripke sheaf incomplete as well, see [3, a Remark in Sect. 9]).

By the way, note that the logics $[\mathbf{Q}\text{-}\mathbf{H}+\mathbf{\Lambda}+D^*]$ and $[\mathbf{Q}\text{-}\mathbf{H}+\mathbf{\Lambda}+D^-]$ are different for every non-classical $\mathbf{\Lambda}$, since it was claimed in [6] that

$$[\mathbf{Q}\text{-}\mathbf{H}+P_2\&J+D^-]\nvdash D^*.$$

Also it was shown in [6] that every I-frame (i.e., every Kripke sheaf) validating $D^-\&J$ validates D^* as well. Here we claim even more:

Proposition. Let $\mathbf{\Lambda}$ be a superintuitionistic propositional logic. Then:
(every I-frame validating $[\mathbf{Q}\text{-}\mathbf{H}+\mathbf{\Lambda}+D^-]$ validates D^*) iff $\mathbf{\Lambda}\vdash\delta J$.

Here $\delta J=[\,Q_0\vee(Q_0\to\neg Q\vee\neg\neg Q)\,]$
(this formula states that J holds in all strictly future worlds: 'tomorrow J').

2 The proof of Main Lemma uses the functor semantics, see [1].

Namely, let $\mathcal{C}$ be a category with a frame representation W; this means that $W=Ob(\mathcal{C})$ is the set of objects of $\mathcal{C}$ pre-ordered by the following relation:

$u \le v$ iff $\mathcal{C}(u,v) \ne \varnothing$, i.e., iff in $\mathcal{C}$ there exists a morphism from u to v.

A $\mathcal{C}$-*set* (a SET-valued functor, or a presheaf over $\mathcal{C}$, inhabited, i.e., with non-emptiness assumption) is a triple $\mathbb{F} = (W, \overline{D}, \overline{E})$, in which $\overline{D} = (D_u : u \in W)$ is a family of disjoint non-empty domains and $\overline{E} = (E_\mu : \mu \in Mor(\mathcal{C}))$ is a family of functions with $E_\mu : D_u \to D_v$ whenever $\mu \in \mathcal{C}(u,v)$ (i.e., μ is a morphism from u to v), satisfying the usual requirements:

$E_{\mu\circ\mu'} = E_{\mu'} \circ E_\mu$ for $\mu \in \mathcal{C}(u,v)$, $\mu' \in \mathcal{C}(v,w)$
(i.e., $E_{\mu\circ\mu'}(a) = E_{\mu'}(E_\mu(a))$ for any $a \in D_u$), and
$E_{1_u} = 1_{D_u}$ (the identity function on D_u corresponds
to the identical morphism $1_u \in \mathcal{C}(u,u)$, $u \in W$).

A *valuation* $u \vDash A(\mathbf{a})$ (for $u \in W$ and $\mathbf{a} = (a_1, \dots, a_n) \in (D_u)^n$) in $\mathbb{F}$ satisfies the monotonicity:

$$u \vDash A(\mathbf{a}) \Rightarrow v \vDash A(E_\mu(\mathbf{a})) \quad \text{for } u \le v \text{ and } \mu \in \mathcal{C}(u,v),$$

and the usual inductive clauses for connectives and quantifiers, e.g.

$u \vDash (B \to C)(\mathbf{a}) \Leftrightarrow \forall v \ge u \, \forall \mu \in \mathcal{C}(u,v) \, [\, v \vDash B(E_\mu(\mathbf{a})) \Rightarrow v \vDash C(E_\mu(\mathbf{a})\,]$,
$u \vDash \forall x B(\mathbf{a}, x) \Leftrightarrow \forall v \ge u \, \forall \mu \in \mathcal{C}(u,v) \, \forall c \in D_v \, [\, v \vDash B(E_\mu(\mathbf{a}), c)\,]$,
etc. (here we write $E_\mu(\mathbf{a}) = (E_\mu(a_1), \dots, E_\mu(a_n))$ for $\mathbf{a} = (a_1, \dots, a_n)$).

A formula $A(\mathbf{x})$ is *valid* in a $\mathcal{C}$-set $\mathbb{F}$ if it is true w.r.t. all valuations in $\mathbb{F}$, i.e., if $u \vDash A(\mathbf{a})$ for all $u \in W$ and $\mathbf{a} \in (D_u)^n$. The *predicate logic of a* $\mathcal{C}$-*set* $\mathbb{F}$ is the set

$$\mathbf{L}(\mathbb{F}) = \{\, A \mid \text{all substitution instances of } A \text{ are valid in } \mathbb{F}\,\};$$

note that the set of formulas valid in $\mathbb{F}$ in general is not substitution closed (cf. e.g. [3, Remark in Sect. 5]).

Lemma. Let $\mathcal{C}$ be a category with one object 0 and two arrows: 1_0 and μ_0.
Let $\mathbb{F}$ be a $\mathcal{C}$-set with two-element domain $D_0 = \{a, b\}$, $E_{\mu_0}(a) = E_{\mu_0}(b) = b$.
Then $P_2 \& J \& D^* \in \mathbf{L}(\mathbb{F})$, and $K \notin \mathbf{L}(\mathbb{F})$.

To conclude, let us mention the following well-known

Fact. Let W_0 be a (rooted) poset. Then the following conditions are equivalent:
(1) $K \in \mathbf{L}(M)$ for every Kripke frame M based on W_0,
(2) $K \in \mathbf{L}(M)$ for every I-frame M based on W_0, and
(3) W_0 satisfies the *McKinsey property*: $\forall u \, \exists v \ge u \, [v \text{ is maximal in } W_0]$.

On the other hand, K is valid e.g. in every frame M with a finite constant domain (based on an arbitrary W).

Now we can give a counterpart to this fact for the functor semantics:

Claim. Let $\mathcal{C}_0$ be a category. Then the following conditions are equivalent:
(1) $K \in \mathbf{L}(\mathbb{F})$ for every $\mathcal{C}_0$-set $\mathbb{F}$, and
(2) $\forall u \, \exists v \ge u \, [\forall w \ge v \, (v \ge w)]$ (i.e., v lies in a maximal cluster of the frame representation W_0 of $\mathcal{C}_0$) and
for every v in a maximal cluster of W_0: $\forall \mu \in \mathcal{C}(v,v) \, \exists \mu' \in \mathcal{C}(v,v) \, [\, \mu \circ \mu' = 1_v\,]$.

Note that every frame $M = (W, U)$ (with a finite constant domain) can be represented as an isomorphic $\mathcal{C}$-set (for any category $\mathcal{C}$ based on W).

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